Arc Length Based Maximal Lyapunov Functions and Domains of Attraction Estimation for Polynomial Nonlinear Systems

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Abstract

In the phase space of a dynamical system containing an asymptotically stable equilibrium point, the Arc Length Function (ALF) is defined as sum of length differential elements of phase trajectories starting from state points and ending at the equilibrium point. It is shown that receding from the origin and verging on the Domain of Attraction (DOA) boundary causes ALF to be increase drastically. According to the latter issue and regarding to other properties, it is shown that ALF is a *Maximal Lyapunov Function*. To this end, a numerical method to approximate the ALF as a polynomial function is proposed. To ensure that the approximated function has positive value and negative derivative inside the desired region, the homotopy continuation method is used. Thus, the approximated ALF presents an ensured Lyapunov behavior to estimate DOA. The efficacy of the proposed method is demonstrated by several simulation examples.

1 Introduction

Stability is one of the most crucial issues in nonlinear control systems, both in analysis and design [14]. The domain of attraction (DOA) of an equilibrium point plays a key role in understanding stability. For example, some knowledge of the size and shape of the DOA is usually required for the proper planning of the operation of the nonlinear system, such as controller design or stability analysis [25]. In [1] it has been asserted that determining the stability region of an equilibrium point for a nonlinear system is a significant task in applications, such as electric power systems [35], economics [3], and ecology [33].

Unfortunately, determination and accurate representation of a general DOA is currently cumbersome. Therefore, estimating the DOA with simple shapes

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has been an essential tool in control system analysis. During the last few years, many approaches have been introduced in the literature in order to deal with the problem of DOA estimation. The majority of these solutions are built on important theoretical results. However, there is serious concern regarding their practicality. It has been demonstrated that many of the suggested algorithms, because of dependency of computational size and dimension of systems, are either impractical for engineering applications or incompatible with numerical implementations, especially when the order of the system is high [43]. Consequently, efforts have been made for estimating the domain of attraction during the recent years. Generally, these attempts can be divided into two major categories.

The first category for DOA estimation originated from Lyapunov theory of stability such as [17, 37, 40] and a number of earlier approaches like [30] which take advantage of quadratic functions. A key contribution to this category was Zubov's method [47, 48], providing necessary and sufficient conditions for asymptotically stable regions around stable equilibrium points.

Chesi and Hachicho, in [10, 12, 21], show how the problem of finding DOA for polynomial systems can be translated to a basic optimization problem. Also in [13], powerful LMI (Linear Matrix Inequalities) software packages are successfully used to solve an optimization problem for non-polynomial systems in order to estimate DOA. Moreover, some methods based on sum-of-squares (SOS) relaxations are introduced which lead to a convex optimization problem constrained by BMI (Bilinear Matrix Inequalities) [41, 39, 36, 22]. Likewise, in [32], a global optimization to estimate DOA of maximal Lyapunov functions is used. The methods presented in [42] are proposed to compute Lyapunov functions and obtain an estimation of DOA given by the level set of these functions. Some applications are in nonlinear models of electrical machines such as [16] and [44]. In [23], the theory of moments is applied to approximate the DOA of uncertain polynomial systems (uncertainty in the coefficients).

Using these methods, a Lyapunov function could be provided as a stability proof, as well as a measure of the DOA in which Lyapunov function decreases along the flow. Additionally, there are techniques for finding estimates of DOA in uncertain polynomial systems. For example, for polynomial systems with a significant level of uncertainty, Cruck and Seube used discretization in time to drift constant sets backwards along the flow of the vector field [17]. Also in [15] and [11], through rational Lyapunov functions and SOS techniques and LMI methods, robust DOA is estimated for systems which have uncertainty in parameters. In order to obtain an approximate measure of DOA, Prakash and Soma employed quantifier elimination method by partial cylindrical algebraic decomposition to determine Lyapunov functions [37]. Another approach is a multi dimensional grid-based method relying on the use of Chebychev points and the theorem of Ehlich and Zeller [18, 38]. The second category consists of numerical approaches for determining the optimum Lyapunov function. Alternate DOA estimation methods can be summarized as iterative computational algorithms, used to determine approximate measures of the region of attraction on the basis of the backward evolution of the dynamical system. These are discussed in, for example, [19, 20, 34, 46, 27]. Such algorithms begin with a limited number of points near the equilibrium point and continue by transferring them backwards in time. This approach is quite useful for two dimensional systems, but not beneficial enough for systems with more dimensions due to the curse of dimensionality. In fact, tracking of a limited group of points may not be sufficient to accurately characterize the system behavior owing to nonlinearity. This problem can be addressed by looking for analytical expressions for the DOA's boundary. Thus, the shape factor concept introduced in [24] is employed to find a polynomial function estimating DOA in [26].

The primary focus of this paper is unforced independent dynamical systems, represented as (1). The main reason for this limitation is that many phenomena and processes in real world can be precisely modeled by independent dynamical systems [21]. Also, as previously mentioned, there is currently no perfect procedure for constructing Lyapunov functions. Thus, in most Lyapunov based methods in literature, quadratic Lyapunov functions are employed to estimate DOA, but they lead to a poor DOA estimation in that the estimated DOA demonstrate a elliptical representation of the DOA, while the DOA might be more complex. Hence, the main contribution of this paper is to introduce a procedure for constructing Lyapunov functions by approximating the ALF (properties of which are introduced in [5]) and estimating the DOA by the direct Lyapunov DOA estimation method. In Section 2, ALF is explained in more detail. Specifically, it is shown that ALF is a Lyapunov function inside the DOA of the system. Subsequently, the ALF will be employed for estimating the DOA for polynomial non-linear systems. In Section 3, an approximation of ALF is presented via a parametric model, which then allows estimation of DOA which the proposed procedure is based on enlarging the DOA. One of the computational tools this paper relies on is *homotopy continuation* [45], a well known method for numerically solving polynomial equation systems, where the theory and computational tools are collectively as numerical algebraic geometry (NAG). NAG is used across science and engineering [28, 8]. In this regards, Bertini [4], one of the powerful numerical solvers implementing homotopy continuation, and Bertini_real [7], a tool for computing real curve and surface decompositions [31]. In this paper, Bertini and Bertini_real are used to verify that the approximated ALF has quasi-Lyaponuv properties. In Section 4, the concept of ALF is employed to estimate the DOA with some examples illustrating our DOA estimation method. We conclude in Section 5.

2 Approximating the Arc Length Function

Arc Length Function (ALF) is introduced in [5], and ALF stability and relevant theories are discussed in [6]. To the best of our knowledge, there is no procedure to approximate ALF. Since the characteristics for so-called Maximal Lyapunov in [44] is same to the ALF's features, the ALF is a maximal Lyapunov function. In such functions, the "blow up" effect of denominator near the boundaries of the DOA implies that their level sets closely represent the region of stability in that part of state space. In [44], a recursive procedure is proposed to simultaneously construct the maximal function and compute the estimation of the DOA.

The approach in this paper is to find polynomial functions as ALF instead of rational functions. The main reason for the choice is simplicity of constructing a polynomial function with the Least Square method. In fact, the "blow up" effect is not present in ALFs and based on this limitation, iso-length contours are used to approximate ALF which are not near the boundaries of the DOA. With help of theorems which will be subsequently presented, and ALF's properties in DOA, an approximation of ALF is introduced. Later in Section 3, an estimation of DOA is achieved using the approximated ALF.

2.1 Arc Length Function

Our main subject of study is an unforced dynamical system, represented as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathbb{R}^n \quad \boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}, \tag{1}$$

where $\boldsymbol{f} : D \rightarrow \mathbb{R}^n$ is continuous derivative in $D \subset \mathbb{R}^n$, $\boldsymbol{f}^\top = [f_1\, f_2 \ldots f_n]$, and $\boldsymbol{x}^\top = [x_1\, x_2 \ldots x_n]$. It is assumed that the origin is an asymptotically stable equilibrium point of the system. Let $\Omega \subset \mathbb{R}^n$ denote its domain of attraction. The length element for phase trajectory at moment of τ is defined as

$$dl(\tau) = \sqrt{dx_1^2(\tau) + dx_2^2(\tau) + \ldots. + dx_n^2(\tau)}, \tag{2}$$

$$dx_i(\tau) = f_i((x(\tau)))d\tau. \tag{3}$$

where $d\tau$ denotes the time differential element.

The phase trajectory length function of the system defined in (1), for any point of $\boldsymbol{x}(t) \in \Omega$, is denoted by $L(\boldsymbol{x}(t))$. which is equal to sum of length differential elements on phase trajectory curve from $\tau = t$ to $\tau = \infty$. In other words:

$$L(\boldsymbol{x}(t)) = \oint_{\tau=t}^{\tau=\infty} dL(\tau) \tag{4}$$

From (1), it can be concluded that if we are at the origin at t, phase trajectory length will be zero at all times:

$$L(\boldsymbol{0}) = 0. \tag{5}$$

According to (1), (2), and (4), length function derivative can be written as follows:

$$L(\boldsymbol{x}(t)) = \oint_{\tau=t}^{\infty} \|\boldsymbol{f}(\boldsymbol{x}(\tau))\| d\tau \quad \text{with} \quad \|\boldsymbol{f}(\boldsymbol{x}(\tau))\| = \sqrt{\boldsymbol{f}^T(\boldsymbol{x}(\tau)) \cdot \boldsymbol{f}(\boldsymbol{x}(\tau))}. \tag{6}$$

From (6) is seen that the length function is a positive definite function. Furthermore, one can show that derivative of the length function is negative definite.

$$L(\boldsymbol{x}(t)) = \int_{\tau=t}^{\tau=t+\Delta t} \|\boldsymbol{f}(\boldsymbol{x}(\tau))\| d\tau + \int_{\tau=t+\Delta t}^{\infty} \|\boldsymbol{f}(\boldsymbol{x}(\tau))\| d\tau. \tag{7}$$

If $\Delta t \to 0$,

$$L(\boldsymbol{x}(t)) = \|\boldsymbol{f}(\boldsymbol{x}(t))\|\Delta t + L(\boldsymbol{x}(t + \Delta t)). \tag{8}$$

Therefore, by supposing $\boldsymbol{x}(t) \neq \boldsymbol{0}$ and $\Delta t \to 0$, we have

$$\dot{L}(\boldsymbol{x}(t)) = \lim_{\Delta t \to 0} \frac{L(\boldsymbol{x}(t + \Delta t)) - L(\boldsymbol{x}(t))}{\Delta t} = -\|\boldsymbol{f}(\boldsymbol{x}(\tau))\| < 0. \tag{9}$$

Since in (1) $\boldsymbol{f}(\bullet)$ is a continuous derivative function and according to (6) and (9), it is concluded that ALF is a continuous derivative function, too. Furthermore, from (5), (6), and (9), one can conclude that ALF is a Lyapunov function in Ω.

Definition 1: *Assume the system can be initialized at each state $\boldsymbol{x} \in \Omega$ at t=0 . For each $c > 0$ an* **iso-length set** *is defined as follows:*

$$\Gamma_c = \{\boldsymbol{x} \in \Omega \,|\, L(\boldsymbol{x}) = c\} \tag{10}$$

That is, the iso-length set Γ_c is the set of state points whose ALF is c.

Theorem 1 *In system (1), for any bounded $c > 0$ the iso-length set Γ_c is not null.*

Proof: Since the origin is asymptotically stable $\Omega \neq \varnothing$ and there is a $\bar{c} > 0$ that $\Gamma_{\bar{c}}$ is not null. Assume $\bar{c}$ is the greatest bounded value that $\Gamma_{\bar{c}} \neq \varnothing$. It is readily to understand for each $\boldsymbol{x}(t) \in \Gamma_{\bar{c}} : L(\boldsymbol{x}(t)) = c$ and since the system in Ω has only one equilibrium point, for any $t_1 < t, \boldsymbol{x}(t_1) \neq \boldsymbol{x}(t)$ and regarding to (9):

$$L(\boldsymbol{x}(t)) = \int_{\tau=t_1}^{\tau=t} \|\boldsymbol{f}(\boldsymbol{x}(\tau))\| \, d\tau + \int_{\tau=t}^{\infty} \|\boldsymbol{f}(\boldsymbol{x}(\tau))\| \, d\tau \tag{11}$$

$$L(\boldsymbol{x}(t)) = \int_{\tau=t_1}^{\tau=t} \|\boldsymbol{f}(\boldsymbol{x}(\tau))\| \, d\tau + \bar{c}$$

Since $\boldsymbol{f}(\boldsymbol{x}(\tau)) \neq 0$, $(\boldsymbol{x}(t_1)) > \bar{c}$. Consequently, $\bar{c}$ is not the greatest bounded value for which $\Gamma_{\bar{c}} \neq \varnothing$ and hence for any bounded $c > 0$ the iso-length set $\Gamma_{\bar{c}}$ is not empty. ■

Remark 1 (Maximal Lyapunov Function) *By verging on the edge of DOA boundary, the ALF value converges to infinity. Predicated on this fact and considering that the ALF is a Lyapunov function, it can be found out the ALF inherited all properties of the Maximal Lyapunov Function. Based on this matter and the aforementioned features for Maximal Lyapunov function in [44], the ALF can represent a larger DOA for dynamic nonlinear systems.*

Also, as preliminary definition for the next sections, Hessian operator is defined as below:

$$H(G(\boldsymbol{x})) = \frac{1}{2} \frac{\partial^2 G(\boldsymbol{x})}{\partial \boldsymbol{x}^2} \Big|_{\boldsymbol{x}=\mathbf{0}} \tag{12}$$

2.2 Approximation of ALF

In this section, ALF for system (1) is approximated by a numerical method. For this purpose, two procedures are considered. In the first procedure, by discretization of the system (1) and equation (6), ALFs are computed for all considered state points, which are nodes of a grid around the origin. This procedure is the main procedure to assign length property to state space points with a numerical method. In the subsequent procedure, using the set of state points and their corresponding length, a parametric model is approximated through Least Squares (LS) technique. According to the the remark 2.1, ALF is a kind of the maximal Lyapunov functions. Although in [44] rational function structure is assumed as appropriate form for the maximal Lyapunov functions, in this paper, for the sake of using LS, the polynomial form with a certain degree like $V(\boldsymbol{x})$ as truncated structure of the rational function is used. Because one does not have *a priori* knowledge about the correct degree of the ALF, the following three constraints must be satisfied by approximated ALF:

1. Eigenvalues of $H(V(\boldsymbol{x}))$ entirely have positive real part to ensure that there is a certain positive area around the origin for approximated V.
2. Eigenvalues of $H(\dot{V}(\boldsymbol{x}))$ entirely have negative real part to ensure that there is a certain negative region around the origin for $\dot{V}$.
3. Bertini and Bertini_real satisfies the condition that there exist no other roots inside a certain area around the origin for V and $\dot{V}$.

In Step 1, by allocating suitable parameters, the desired accuracy and speed of calculation are regulated. In Step 2, with a simple search within the desired area around the equilibrium point, all intended points which have a trajectory path into the origin are detected. Length of each point which is converged to origin calculated as well. Now, all the points located in estimated DOA have length properties. So, with respect to these properties, iso-length contour could be display. With a simple least square for these contours and with considering $L(\boldsymbol{0}) = 0$, an estimate for the ALF as a Lyapunov function can be presented for (1). To this end, considered polynomial function includes no linear term in its structure. As previously mentioned, iterating over degree of ALF allows to select the one maximizing the size of the computed DOA. A procedure to do so is presented in Algorithm 2.

Procedure 2 is used to determine the best order of ALF. Initial parameters such as maximum-minimum degree of polynomial will be set. For each approximated ALF with degree between n_{min} and n_{max}, lines 5-12 will be executed.

In the line 5, $H(V(\boldsymbol{x}))$ and $H(\dot{V}(\boldsymbol{x}))$ symmetric matrix related to coefficients of quadratic terms of approximated ALF and time derivative of this function, respectively will be obtained. To reduce calculation load, the defined condition $Eigenvalues(H(V(\boldsymbol{x}) > 0)$ and $Eigenvalues(H(\dot{V}(\boldsymbol{x}))) \leq 0)$ are computed as preliminary conditions to calculate subsequent lines. Since the approximated ALF does not have linear part, these constraints ensure a positive definite region for the approximated $V(\boldsymbol{x})$ which it contains a negative definite

Algorithm 1 A Grid scanning procedure to compute ALFs and function approximation of ALF

1: **Step1:** Initialize parameters:
Δt: Time variation for discretization of system (1) and equation (6).
Δx: Distance between adjacent nodes on a grid.
t_{max}: A threshold of maximum time used to ignore trajectories which have not been neared to origin before a certain time.
r_0: A threshold of minimum distance used to terminate the length computation procedure for a certain state.
2: **Step2:** Initialize a set of state points (nodes) $\boldsymbol{x}^i{}_{(i=1)^N}$ as nodes of a grid with distance Δx around the equilibrium point.
3: **Step3:** Compute ALF for each node of the grid.
4: **for** i=1:N **do**
5:     set $t = 0$, $\boldsymbol{x} = \boldsymbol{x}^i$, $R = \{\}$ and $L = 0$
6:     **while** $t \leq t_{max}$ or $\| \boldsymbol{x}_i \| < r_0$ **do**
7:         calculate time $t \leftarrow t + \Delta t$
8:         calculate $\boldsymbol{f}(\boldsymbol{x})$ for system (1).
9:         recalculate $\boldsymbol{x}$ through discretized of equation (1):
10:         $\boldsymbol{x} \leftarrow \boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x})\Delta t$
11:         calculate L through discretized of equation (6):
12:         $L \leftarrow L + \|\boldsymbol{f}(\boldsymbol{x})\|\Delta t$
13:     **end while**
14:     **if** $(t < t_{max})$ **then**
15:         $L(\boldsymbol{x}^i) \leftarrow L$
16:         $R \leftarrow \{R, (\boldsymbol{x}^i, L(\boldsymbol{x}^i))\}$
17:     **end if**
18: **end for**
19: In order to estimate iso-length sets Γ_{c_1} , $\Gamma_{c_2}, \ldots,$ Γ_{c_d} use data set R and divide them to d classes of states having near enough lengths respectively to c_1,c_2 ,...,c_d. (c_1,c_2 ,...,c_d can chosen respected to resulted $L(\boldsymbol{x}^i)$ in R).
20: Use data set R and identify a parametric model $\hat{L}(x)$ by curve fitting technique, in order that $\hat{L}(\boldsymbol{x})$ equivalence to $L(\boldsymbol{x})$ with LS error.

area for $\dot{V}(\boldsymbol{x})$ around the origin analytically. To attain this end, the proposed procedure is benefited by analyzing the quadratic terms of those functions to check aforementioned features. Thus, the eigenvalues of $H(V(\boldsymbol{x}))$ and $H(\dot{V}(\boldsymbol{x}))$ will be checked before calculating the hyper-volume of the corresponding DOA and Bertini check.

In 6th line, DOA's hyper-volume according to approximated ALF denoted by ν will be calculated with a simple numerical optimization approach. Alternatively, a global optimization approach [32] can be used to calculate the hyper-volume. In the Bertini check section, Bertini software is run to produce the Numerical Irreducible Decomposition. It produces the witness points, the points on the curve or surface, which will be subsequently tracked around dur-

Algorithm 2 Order determination for computing optimal ALF

1: Set initial parameters:
2: Set n_{min} and n_{max} as minimum and maximum polynomial order of ALF and consider $n = n_{min}$.
3: **while** $n \leq n_{max}$ **do**
4:     Calculate ALF polynomial function with nth order.
5:     **if** $Eigenvalues(H(V(\boldsymbol{x}))) > 0 \;\; \&\& \; Eigenvalues(H(\dot{V}(\boldsymbol{x}))) \leqq 0$ **then**
6:         Calculate corresponding hyper-volume of estimated DOA $(\nu_{V(\boldsymbol{x})})$.
7:         **if** $BertiniCheck == Ok$ **then**
8:             Save $\nu_{V(\boldsymbol{x})}$.
9:         **end if**
10:     **end if**
11:     $n = n + 1$
12: **end while**
13: Select $V(\boldsymbol{x})$ corresponding to maximum of $\nu_{V(\boldsymbol{x})}$.

ing Bertini_real. It also produces the witness linears which are used in the regeneration and slicing steps of the algorithm. Considering gathered required files, Bertini_real is used to ensure that both the candidate functions and their derivatives have no roots around the origin, except the equilibrium point itself. Finally in the 8th line, the optimal ALF corresponding to the maximum hyper-volume will be selected among the candidate functions.

3 Estimating the Domain of Attraction

In this section, by considering the approximated ALF, an estimation for DOA of system (1) will be introduced.

There are many methods for estimating the stability regions about asymptotically stable equilibrium points. The proposed procedures have the advantage of being used for either bounded or unbounded analysis region state space. It is worth noting that the boundary of DOA is not considered as a part of the DOA. That is, the DOA is treated as open. The reason is that on the boundary of DOA the length does not necessarily converge to a finite value due to possibility of limit cycle or separatix phase. So, the arc-length value in the exact boundary of DOA will be increase drastically. In this paper, approximated ALF is employed in direct Lyapunov function method to estimate the DOA.

3.0.1 Estimation of DOA Through Direct Lyapunov Function Method

The DOA can be estimated through Lyapunov functions, the fundamental theorem of which can be formulated as follows.

Theorem 2 *Let $V(\boldsymbol{x})$ be a Lyapunov function for (1). The DOA can be esti-*

mated as

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^n, 0 < V(\boldsymbol{x}) < c, \dot{V}(\boldsymbol{x}) < 0\}, \tag{13}$$

where c is a positive real constant. Then Ω contained in the DOA of $\boldsymbol{x} = 0$.

The proof is based on the fact that the conditions on Ω certify that any solution path starting from Ω strictly and monotonically converges to the origin [29]. It is important to mention that the theorem gives only a simple estimation, and that the DOA can actually be much larger than Ω. The theorem gives no information about how to choose or create the function V. Constructing a quadratic Lyapunov function for the linearized system and applying it to the nonlinear system and checking the DOA is a common method. But, these quadratic Lyapunov functions are often poor estimation for the DOA.

In previous sections, ALF was introduced as a Lyapunov function which has positive value and negative derivative in DOA. Hence, ALF is suitable for more accurately estimating of the DOA. In previous section, an approximation method to estimate ALF as a local Lyapunov function has been presented. According to (13), larger level set values c lead to better estimations of DOA. Therefore, estimation of DOA or calculation of the maximum level set of the Lyapunov function can be obtained by solving a problem whose pseudo optimization formulation is as follows:

$$\begin{aligned} &\min_{c,\boldsymbol{x}} \text{ c} \\ \text{s.t.} \quad & V(\boldsymbol{x}) - c = 0 \\ & \dot{V}(\boldsymbol{x}) = 0 \\ & c > 0 \end{aligned} \tag{14}$$

The above optimization problem falls into the category of nonlinear constrained optimization. Since $V(\boldsymbol{x})$ and $\dot{V}(\boldsymbol{x})$ are polynomial, this problem can be solved by adding the tangency constraint introduced in [32]. In this method, $\nabla(V(\boldsymbol{x}) - c) = \epsilon\nabla(d\dot{V}(\boldsymbol{x}))$ constraint will be add to the present constraints. By considering the direct Lyapunov theorem to estimate DOA and contemplating the approximated ALF in previous section, the DOA estimation would be achieved.

4 Illustrative examples

In this section, four dynamical systems are presented to illustrate Procedures 1 and 2. In these examples, an optimal ALF is obtained from polynomial functions with order between 5 and 8 for 2D systems, and between 2 and 5 for 3D systems. These examples are chosen among systems with various properties, such as having a bounded/unbounded DOA, and two- or three- dimensional polynomial systems. For all examples, the guaranteed circle and time variation are considered as $r = 0.01$ and $\Delta t = 0.01$ respectively, and Bertini_real output data for each example are graphically demonstrated. Expansion of the estimated stability domain for mentioned systems and corresponding lower bound for ALF are calculated.

4.1 Example 1 – Vanderpol

Consider the well-known Vanderpol system as an odd polynomial system:

$$E1 = \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -2x_1 - (3 - {x_1}^2)x_2. \end{cases}$$

In this system, there is an unstable limit cycle and the origin is a stable equilibrium. Moreover, the biggest domain of attraction to the origin is the region enclosed by the limit cycle. By applying method explained in previous sections and supposing $dx = 0.04$, iso-length contours are obtained. It is worth to mention, increasing the value of length leads to converge the contours to a compressed area at the edge of DOA. In this area, just by increasing the resolution of computation, all points close to the length contours will be computable. To approximate ALF, proposed procedures are applied and obtained results are presented in Table 1.

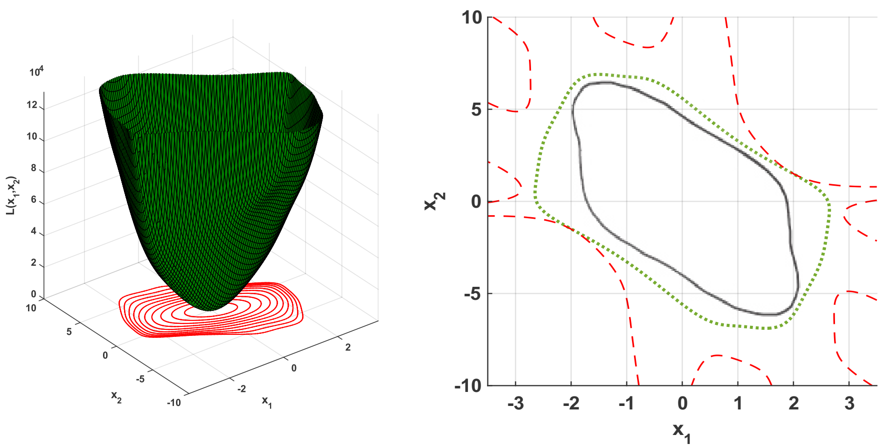
Table 1: Results of applied procedures for E1

Degree	Eigenvalues(H_{V_x})	Eigenvalues($H_{\dot{V}_x}$)	ν_i
5	{0.25,1.38}	{-2.41,-0.82}	32.01
6	{0.33,1.60}	{-2.73,-1.06}	40.02
7	{0.33,1.60}	{-7.98,-0.32 $\times 10^{-3}$}	39.48
8	{0.31,5.10}	{-16.48,-0.76}	47

In Fig. 1a the approximated ALF (8th order polynomial) and its contours are presented. The approximated ALF is:

$$\begin{aligned} V_{E1}(x_1, x_2) = & -45x_1^8 - 38x_1^7x_2 + 0.17x_1^7 + 11x_1^6x_2^2 - 0.36x_1^6x_2 + 944x_1^6 + 92x_1^5x_2^3 - \\ & 0.58x_1^5x_2^2 + 688x_1^5x_2 - 0.28x_1^5 + 82x_1^4x_2^4 - 0.71x_1^4x_2^3 + 588x_1^4x_2^2 + \\ & 7.9x_1^4x_2 - 5.510^3x_1^4 + 33x_1^3x_2^5 - 0.44x_1^3x_2^4 + 133x_1^3x_2^3 + 7.4x_1^3x_2^2 - \\ & 3.9 \times 10^3x_1^3x_2 + 3.6x_1^3 + 7.3x_1^2x_2^6 - 0.15x_1^2x_2^5 + 4.2x_1^2x_2^3 - 2.4 \times 10^3x_1^2x_2^2 \\ & - 16x_1^2x_2 + 2.2 \times 10^4x_1^2 + 0.81x_1x_2^7 - 0.03x_1x_2^6 - 0.88x_1x_2^5 + 1.2x_1x_2^4 \\ & - 711x_1x_2^3 - 9.8x_1x_2^2 + 1.2 \times 10^4x_1x_2 + 0.025x_2^8 - 2.1 \times 10^{-3}x_2^7 + \\ & 2.3x_2^6 + 0.11x_2^5 - 211x_2^4 - 1.2x_2^3 + 6.9 \times 10^3x_2^2 \end{aligned}$$

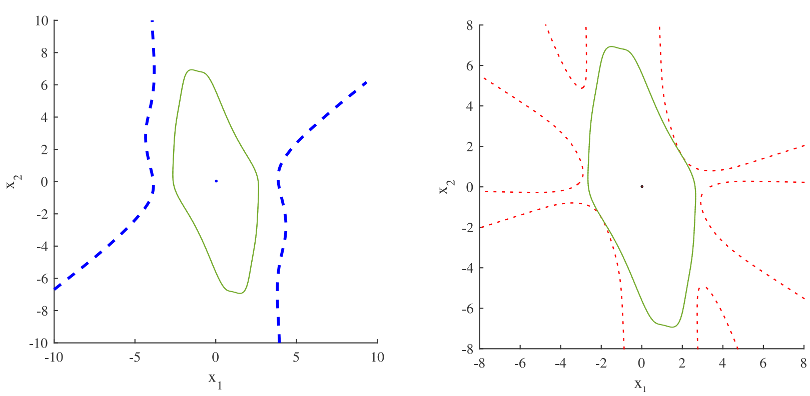
Also, in Fig. 2, Bertini root check results are demonstrated. By considering the corresponding approximated ALF, DOA estimation is presented in Fig. 1b. According to the optimization problem in (14), computed c for optimal ALF is 97445.51.



(a) Approximated ALF with its contours.

(b) Dotted green line and solid black line depict the proposed method and Chesi's results [12], respectively. For red dashed lines, $\dot{V} = 0$.

Figure 1: Approximated ALF and estimated DOA for E1.



(a) Blue dashed curve is $V_{E1} = 0$.

(b) Red dotted curve is $\dot{V}_{E1} = 0$.

Figure 2: Bertini root check results for estimated DOA (Green curve) for E1.

4.2 Example 2 – Two Dimensions, Bounded DOA

Consider an odd polynomial non-linear system with bounded DOA from [9]:

$$E2 = \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1^5 + x_1 x_2^4 + x_1 x_2^2 - 2x_1 + x_2^5 - x_2. \end{cases}$$

The origin is the stable equilibrium point. Initial parameters are considered as $dx = 0.01$. By applying the proposed procedures, following results (Table 2) are obtained.

Table 2: Results of applied procedures for E2

Degree	Eigenvalues(H_{V_x})	Eigenvalues($H_{\dot{V}_x}$)	ν_i
5	{0.75,2.42}	{-3.72,-0.41}	2.96
6	{4.63,1.27}	{-1.26,-0.01}	3.15
7	{1.04,3.85}	{-4.98,-0.62}	3.07
8	{0.40,2.05}	{-2.11,-0.12}	3.03

As it is clear, optimal ALF is 6th order polynomial whose iso-length contours and approximated ALF are presented in Fig. 3a. The approximated ALF can be written as

$$\begin{aligned} V_{E2}(x_1,x_2) = & 3556x_1^6 + 1043x_1^5x_2 + 0.03x_1^5 + 3482x_1^4x_2^2 + 2.058x_1^4x_2 - 3766x_1^4 + \\ & 587.6x_1^3x_2^3 - 0.5779x_1^3x_2^2 - 878.1x_1^3x_2 - 0.06404x_1^3 + 1.32x_1^2x_2^3 - \\ & 1333x_1^2x_2^2 - 2.301x_1^2x_2 + 3947x_1^2 - 493.1x_1x_2^5 - 1.579x_1x_2^4 + 1.073x_2^3 \\ & - 518.1x_1x_2^3 + 1.563x_1x_2^2 + 2863x_1x_2 + 4272x_2^6 - 1.229x_2^5 - 2632x_2^4 + \\ & + 2076x_2^2 \end{aligned}$$

In Fig. 4, Bertini root check results are demonstrated. The corresponding estimate of DOA is represented in Fig. 3b. According to the optimization problem in (14), computed c related to ALF is 3119.59.

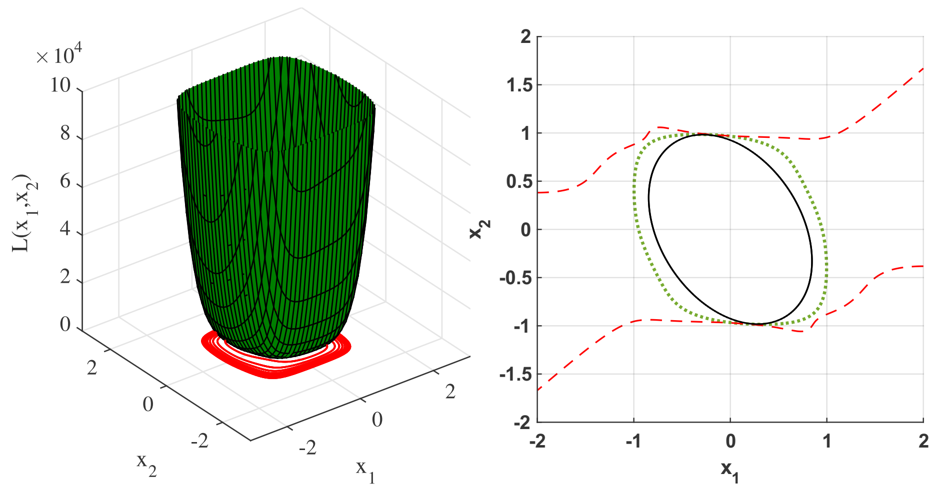
4.3 Example 3 – Two Dimensions Unbounded DOA

Another polynomial system from [2] is considered as below:

$$E3 = \begin{cases} \dot{x}_1 = -50x_1 - 16x_2 + 13.8x_1x_2, \\ \dot{x}_2 = -13x_1 - 9x_2 + 5.5x_1x_2. \end{cases}$$

In this case, the origin is a stable equilibrium point. By considering $dx = 0.05$, the iso-length contours can be obtained. In following, approximated polynomial functions obtained from proposed procedures are provided in Table 3

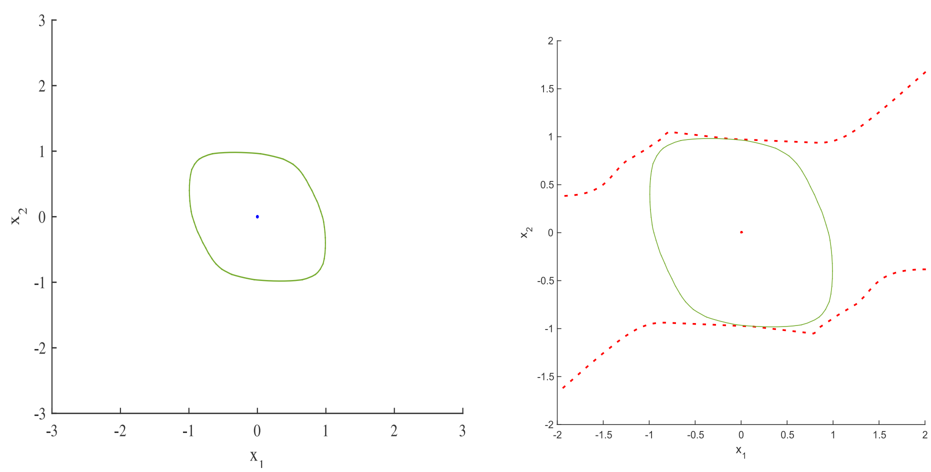
All obtained volumes are larger than the initial volume. Therefore, approximated polynomial with fifth order is considered to be the desired ALF. The contours are presented in Fig. 5a.



(a) Approximated ALF with its contours.

(b) Dotted green line and solid black line depict the proposed method and Chiang's results [16], respectively. For red dashed lines, $\dot{V} = 0$.

Figure 3: Approximated ALF and estimated DOA for E2.



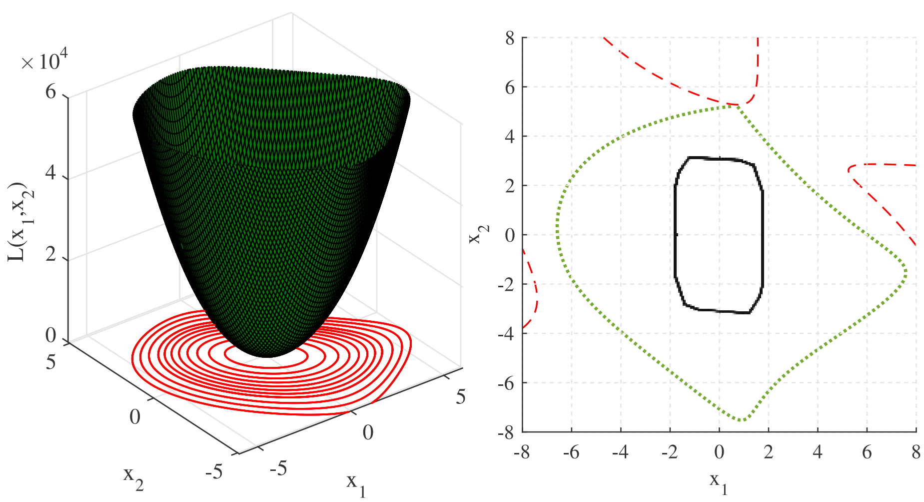
(a) Blue dashed curve is $V_{E2} = 0$.

(b) Red dotted curve is $\dot{V}_{E2} = 0$.

Figure 4: Bertini root check results for estimated DOA (Green curve) for E2.

Table 3: Results of applied procedures for E3

Degree	Eigenvalues(H_{V_x})	Eigenvalues($H_{\dot{V}_x}$)	ν_i
5	{0.69,2.27}	{-160,-25}	133.1
6	{0.2,0.6}	{-40.75,-7.48}	115.67
7	{0.21,0.65}	{-38.58,-8.98}	141.63
8	{0.24,0.74}	{-49.76,-9.37}	137.42



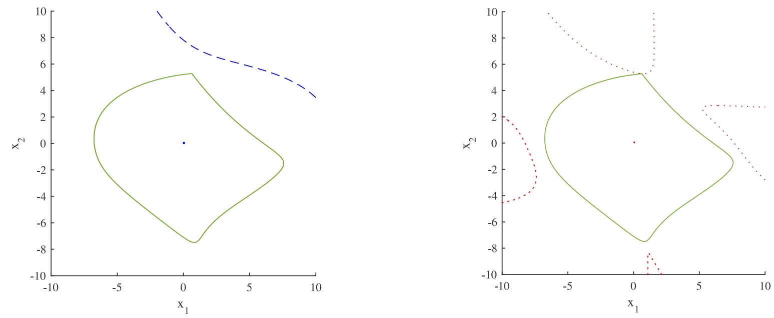
(a) Approximated ALF with its contours.

(b) Dotted green line and solid black line depict the proposed method and Amato's results [2], respectively. For red dashed lines, $\dot{V} = 0$.

Figure 5: Approximated ALF and estimated DOA for E3.

$$
\begin{aligned}
V_{E3}(x_1, x_2) = & -0.27x_1^5 - 3.067x_1^4x_2 - 22.06x_1^4 - 3.151x_1^3x_2^2 - 35.64x_1^3x_2 + \\
& 30.12x_1^3 - 16.29x_1^2x_2^3 - 20.72x_1^2x_2^2 + 398.9x_1^2x_2 + 2942x_1^2 - \\
& 19.4x_1x_2^4 - 125x_1x_2^3 + 452.2x_1x_2^2 + 3519x_1x_2 - 4.942x_2^5 - \\
& 56.83x_2^4 + 207.3x_2^3 + 4208x_2^2
\end{aligned}
$$

Estimated DOA related to approximated ALF is illustrated in Fig. 5b. According to the optimization problem in (14), obtained c for approximated ALF is 82688.30. Also, in Fig. 6, Bertini root check results are demonstrated.



(a) Blue dashed curve is $V_{E3} = 0$.

(b) Red dotted curve is $\dot{V}_{E3} = 0$.

Figure 6: Bertini root check results for estimated DOA (Green curve) for E3.

4.4 Example 4 – Three Spatial Dimensions

This last example is a three-dimensional non-linear polynomial system from [32] as below:

$$E4 = \begin{cases} \dot{x}_1 = -x_1 + x_2 x_3^2 \\ \dot{x}_2 = -x_2 + x_1 x_2 \\ \dot{x}_3 = -x_3 \end{cases}$$

A stable equilibrium point of system is located at the origin. By considering $dx = 0.06$, the iso-length contours can be obtained.

Table 4: Results of applied procedures for E4

Degree	Eigenvalues(H_{V_x})	Eigenvalues($H_{\dot{V}_x}$)	ν_i
2	{0.16,0.17,0.37}	{-0.75,-0.35,-0.32}	29.48
3	{0.19,0.32,0.86}	{-1.72,-0.64,-0.38}	16.20
4	{0.33,0.36,0.42}	{-0.85,-0.72,-0.67}	48.01
5	{0.34,0.40,0.42}	{-0.84,-0.81,-0.68}	61.03

According to Table 4, a fifth order polynomial which possesses maximal volume will been selected as desired ALF. The ALF's contours are illustrated in Fig. 7.

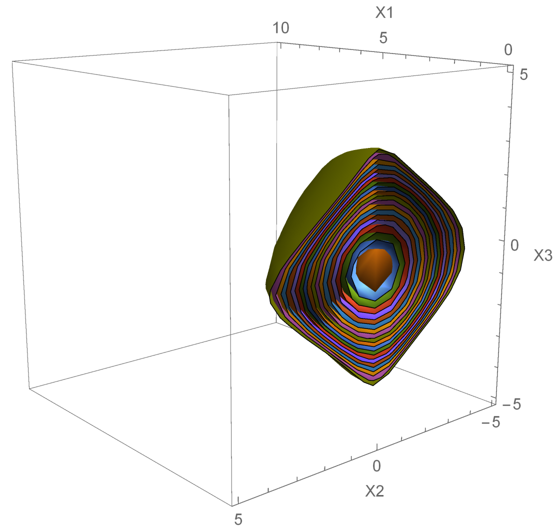


Figure 7: Estimated ALF contours for E4

$$
\begin{aligned}
V_{E4}(x_1,x_2,x_3) = &-31.66x_1^5 + 2.174x_1^4x_2 - 0.1089x_1^4x_3 - 359.1x_1^4 - 28.47x_1^3x_2^2 + \\
&0.5053x_1^3x_2x_3 - 7.657x_1^3x_2 - 54.72x_1^3x_3^2 - 0.95x_1^3x_3 + 1194x_1^3 - \\
&2.339x_1^2x_2^3 + 0.002x_1^2x_2^2x_3 - 227.6x_1^2x_2^2 + 229.3x_1^2x_2x_3^2 + \\
&2.156x_1^2x_2x_3 + 116.9x_1^2x_2 + 0.01473x_1^2x_3^3 - 746.8x_1^2x_3^2 + 23877x_3^2 \\
&+ 23055x_1^2 - 42.57x_1x_2^4 + 0.1638x_1x_2^3x_3 + 44.94x_1x_2^3 + 6.132x_3^3 \\
&+ 0.4822x_1x_2^2x_3 + 1913x_1x_2^2 + 1.887x_1x_2x_3^3 + 1147x_1x_2x_3^2 - \\
&11.99x_1x_2x_3 + 1329x_1x_2 - 18.29x_1x_3^4 + 0.0551x_1x_3^3 + 706.6x_1x_3^2 \\
&- 3.636x_2^5 - 0.01094x_2^4x_3 - 298.5x_2^4 + 89.83x_2^3x_3^2 + 180.9x_2^3 - \\
&0.6144x_2^2x_3^3 + 17.75x_1x_3 + 2351x_2^2x_3^2 + 4.473x_2^2x_3 + 19677x_2^2 + \\
&63.08x_2x_3^4 + 4.943x_2x_3^3 - 1063x_2x_3^2 - 29.05x_2x_3 - 0.1836x_3^5 - \\
&378.3x_3^4 + 0.6051x_1^2x_3 + 578.9x_1x_2^2x_3^2
\end{aligned}
$$

The computed DOA corresponding to approximated ALF is illustrated in Fig. 8. Figure 9 demonstrates the estimated DOA with proposed method is compare with [22]. The computed lower bound is obtained 130432.06. In Fig. 10, Bertini root check results are demonstrated.

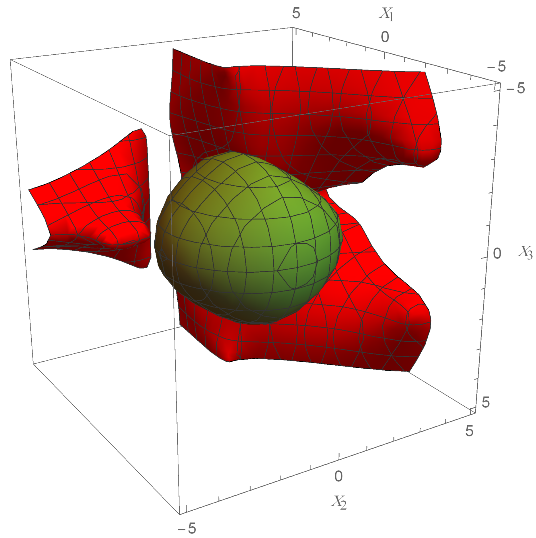


Figure 8: Solid green surface and red surface depict the proposed method and $\dot{V} = 0$ for E4, respectively.

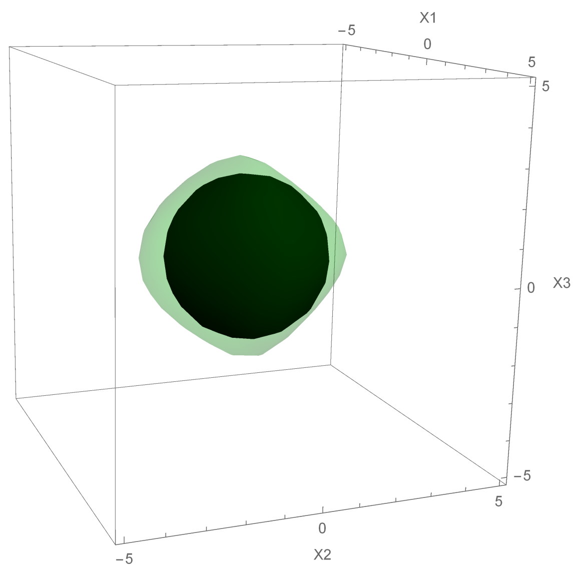
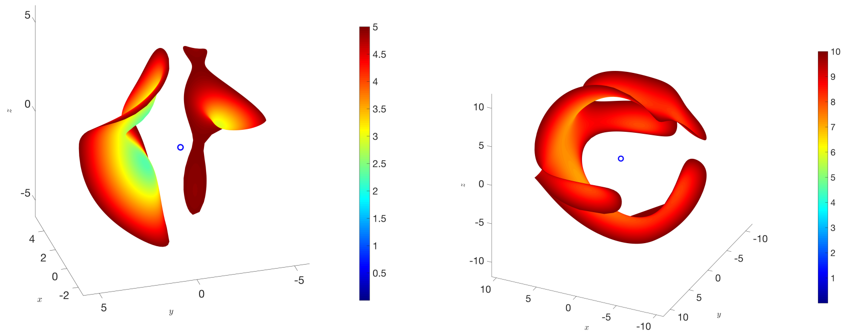


Figure 9: Light green and dark green surfaces depict the proposed method and Hachicho's results [22] for E4, respectively.



(a) $V_{E4} = 0$. (b) $\dot{V}_{E4} = 0$.

Figure 10: Bertini root check results for E4. Colorbar indicates distance from origin – neither surface meets with the origin.

5 Conclusion

In this paper, characteristics of the Arc Length Function (ALF) were studied. It was shown that this function inherits all the characteristics of maximal Lyapunov functions. To approximate ALF, a fast numerical method was suggested to reveal the iso-length contours of non-linear dynamic systems. Using these contours, the LS method was applied to approximate polynomial functions. Also, the homotopy continuation method was used to ensure the ALF follows the Lyapunov manner inside the intended region.

According to direct Lyapunov function and considering significant ALF properties, DOA of some non-linear systems were estimated and the results were illustrated. As compared with the other methods, the proposed method is benefited by both numerical and analytical concepts which leads to achieving larger estimation for DOA. Besides, obtained ALF can be used in analyzing and designing the control systems. The results of proposed method demonstrated for bounded and unbounded DOA and two- and three-dimensional polynomial systems. As a conclusion, this approach together with mentioned theorems achieves better DOA estimates, which could positively affect outcomes in design and analysis of nonlinear control systems.

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