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Waves Interacting on an Annulus, with a Fibonacci style resonance

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Abstract. Building on a DatH report by Ken Monks, we investigate the existence of equilibria in dynamical systems governing the (complex-valued) amplitudes of waves. The waves interact with a Fibonacci-style resonance; that is, there is a recursive structure. Using the Numerical Algebraic Geometry software *Bertini*, we attempt to find the equilibria of the the system, and subsequently classify by stability.

Keywords: Waves, Homotopy Continuation, Equilibria.

1 Introduction

The solving of polynomial systems dates back to antiquity[2]. Certainly the Greeks laid the groundwork for thousands of years of work on the subject, starting with the quadratic formula. The solution of the cubic came in the middle of the second millennium CE, with the thrilling story of Fior, Tartaglia and Cardano. Quartic solutions subsequently were found. Quintic formulae were sought by some of the greatest minds, namely Euler and Lagrange, who failed. Finally Ruffini and Abel proved independently early in the 19th century that no quintic formula exists; a proof of this is commonly found in a graduate level abstract algebra course.

Having firmly established that a formula for the general solution to a polynomial in one variable, of arbitrary degree, does not exist, we turn to our fantastic modern computing machines. Various methods for estimating numerically the solutions to individual polynomials, systems of polynomials, and multivariate polynomial systems, have been developed extensively over the last century.

The solution method of homotopy continuation builds of theorems from topology to track solutions from one system to another as it is continuously deformed. *Bertini* is a software package developed by Dan Bates, Gene Algower, Andrew Sommese, and Charles Wampler, to perform this method numerically. Using predictor-corrector calculations, *Bertini* finds all complex-isolated nonsingular solutions to a multinomial system; it can also detect and sample positive dimensional components of the solution set.

It is important to note that complex space is the algebraic closure of real space, and theorems regarding the existence of solutions to a polynomial invoke the completeness of complex space. Hence, when *Bertini* solves a system, it does so over the complex numbers. However, we (humans) are often interested only in the real-valued solutions to a system. And while the software will find isolated solutions, the real solutions may in fact lie on a complex component, being isolated only over the real numbers. Hence, *Bertini* would miss these.

The dynamical systems studied in this paper are entirely polynomial (or are transformed into such a system). Equilibrium solutions are places where the rate of change is 0; that is, the equilibria of a (polynomial) dynamical system or differential equation are zeros of the polynomial system. *Bertini* is therefore a great candidate for use as a tool with which to numerically find equilibria.

2 The Monks Equations

Determination of the equations

Monks, in [1], explores the D_4 symmetry, in a system of four interacting waves. Mainly, he uses the principle of equivariance to determine the governing equations for such a system, where equivariance states that,

$$\frac{d\gamma\bar{z}}{dt} = \gamma\frac{d\bar{z}}{dt},$$

with γ is a symmetry of the system.

The resonance studied has the four wave vectors $\mathbf{k_0}, \mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}$, with $\mathbf{k_0} + \mathbf{k_1} = \mathbf{k_2}$ and $\mathbf{k_1} + \mathbf{k_2} = \mathbf{k_3}$. There is also the restriction that $|\mathbf{k_0}| = |\mathbf{k_3}|$ and $|\mathbf{k_1}| = |\mathbf{k_2}|$

After asserting the symmetries, nondimensionalizing, and simplifying, Monks determines the following system of equations (now γ is a parameter):

$$\begin{aligned}
\dot{z_0} &= \mu_0 z_0 + \bar{z}_1 z_2 - \gamma z_0(S - |z_0|^2)\\
\dot{z_1} &= \mu_1 z_1 + \bar{z}_0 z_2 + \bar{z}_2 z_3 - \gamma z_1(S - |z_1|^2)\\
\dot{z_1} &= \mu_1 z_2 + \bar{z}_0 z_1 + \bar{z}_1 z_3 - \gamma z_2(S - |z_2|^2)\\
\dot{z_0} &= \mu_0 z_3 + z_1 z_2 - \gamma z_3(S - |z_3|^2),
\end{aligned}$$

where $S = 2\sum_{j=0}^{4} |z_j|^2$. Monks notes that the three parameters μ_0, μ_1, γ must be real, and that $\gamma < 0$.

Equilibrium Solutions

As noted above, *Bertini* will find all complex-isolated solutions to a square polynomial system. To do so for our system of waves, we construct a suitable input file for the program, and solve for random values of the three parameters, μ_0, μ_1, γ, to get a start system. This is sometimes referred to as an 'offline' solve. This start system using a single variable group tracks 81 paths to 81 isolated solutions.

The offline input file is shown below:

```
   INPUT
variable_group z0, z1, z2, z3;
constant mu0,mu1,gamma;
function f1, f2, f3, f4;
mu0 = 0.898107 + 0.283235*I;
mu1 = 0.421873 + 0.1613*I;
gamma = 0.127524 + 0.227002*I;
f1 = mu0*z0+z1*z2-gamma*z0*(2*(z0^2+z1^2+z2^2+z3^2)-z0^2);
f2 = mu1*z1+z0*z2+z2*z3-gamma*z1*(2*(z0^2+z1^2+z2^2+z3^2)-z1^2);
f3 = mu1*z2+z0*z1+z1*z3-gamma*z2*(2*(z0^2+z1^2+z2^2+z3^2)-z2^2);
f4 = mu0*z3+z1*z2-gamma*z3*(2*(z0^2+z1^2+z2^2+z3^2)-z3^2);
END;
```

Then we write an input file for a particular set of parameter values at which we desire to solve, and perform a 'user homotopy', to track the 81 solutions from the random start system to the solutions of our particular system. This is an 'online' solve. The input file for the online run is very similar.

We can perform the online step for an arbitrary set of parameter values. There can be issues arising if the system has positive-dimensional components to the solution set, so we test for such components by performing a 'tracktype1' solve. Indeed, by doing so we verify that there are no positive-dimensional components to the variety, so we can be sure that we have lost none of the real solutions we seek.

Having checked that we are obtaining all solutions, we proceed to solve the system over a mesh of values of the parameters. There is much information to be had, not the least of which is the number of real solutions for a given parameter set. That is, we can construct a kind of bifurcation diagram, indicating regions in parameter space which give different numbers of solutions to the system.

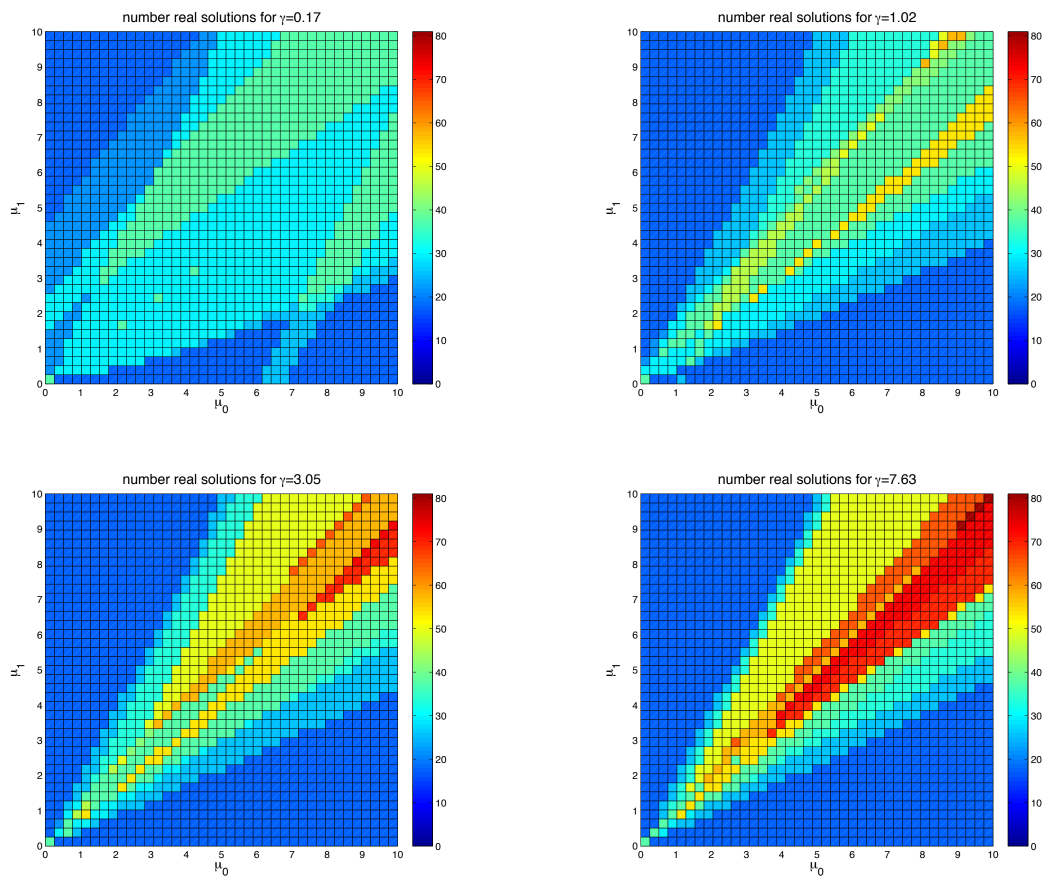

Figure 1: Regions with various numbers of real solutions to the Monks equations in parameter space (μ_0, μ_1, γ). The displayed region is for $(\mu_0\,, \mu_1) \in [0\,, 10] \times [0\,, 10]$. At $\gamma = 0$ there is one solution everywhere, namely the 0 solution; this plot omitted. For low γ, there are few solutions near the parameter-origin, as in (a). As γ increases, regions of higher numbers of solutions appear, and these regions move toward the origin.

In figure 1, we see a series of pseudocolor plots of the number of real solutions at points in parameter space. Omitted from this figure, when $\gamma = 0$, there is exactly one real solution for all values of μ_0, μ_1. Upon increasing γ, we see new regions develop. At first, the wedge-shaped regions have fewer than 10 real solutions. By the time $\gamma = 7.63$, as in the lower-left plot, there is a region which has 81 real solutions.

Having seen that the Monks equations do in fact have nontrivial and multiple solutions, we display the solutions. In figure 2, we show successive zooms into the origin of coordinate space. In particular, this figure is a projection of the entire solution set onto the first two coordinates, z_0, z_1, with the color of the points determined by the value of the fourth coordinate, z_3; no information on these plots comes from z_2. Projections onto other coordinate systems are more or less the same.

One outstanding feature of these plots is the presence of many solutions lying on the coordinate axes; note, too, the clustering of color in the green region, representing $z_3 \approx 0$. In addition, there appear to be separate components to the plot, with perhaps components corresponding to various spaces in coordinate space.

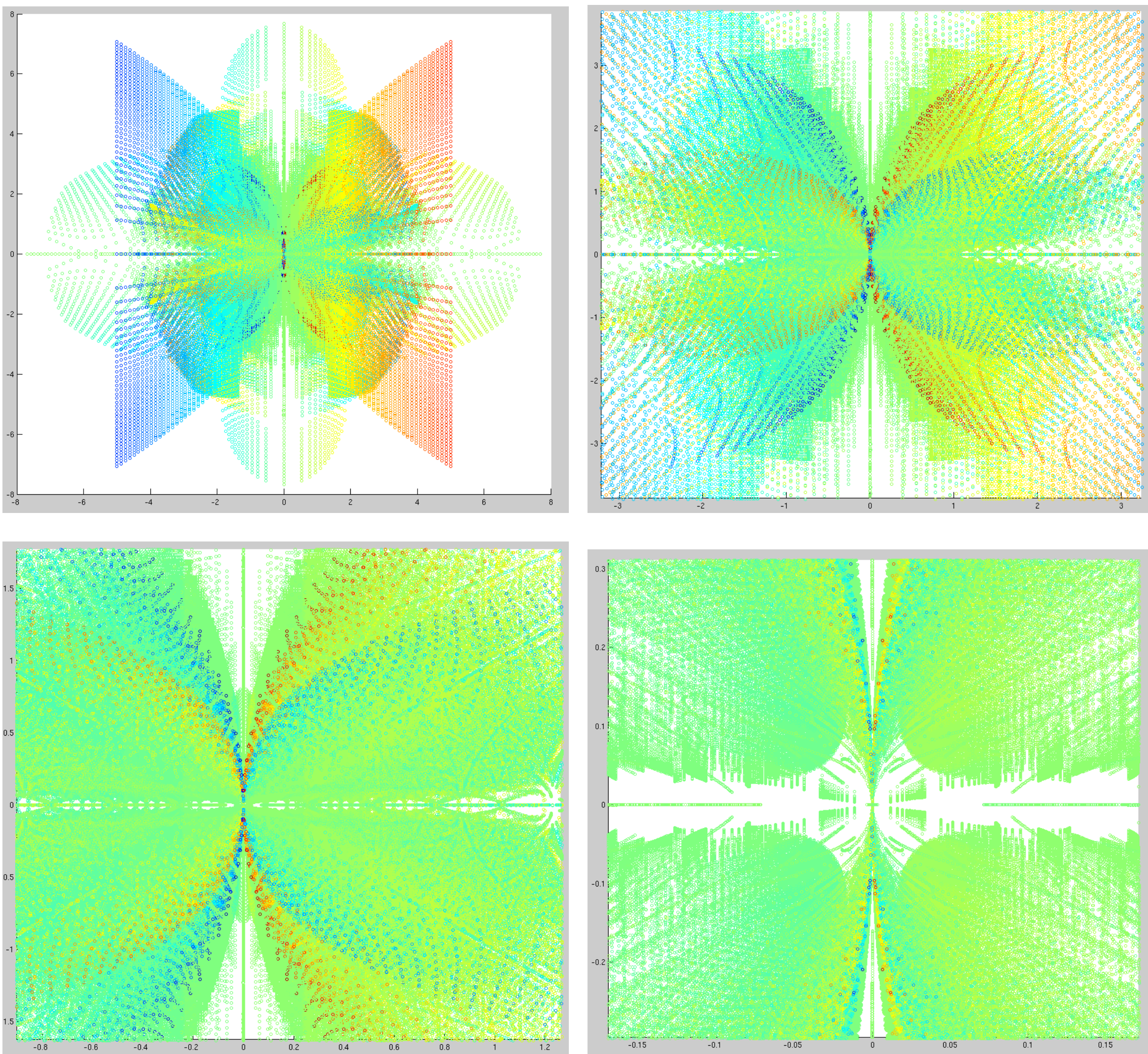

Figure 2: Successive zooms into the origin of coordinate space. Points represent projection onto z_0, z_1, with color determined by z_3. All solutions for the entire scan through parameter space plotted.

Stability

Having equilibrium solutions to the Monks equations, we are then interested in their stability. To determine stability of a solution, we perturb the system linearly around the solution, and drop all nonlinear terms from the expression. Specifically, in our dynamical equations, we perturb in both the real and imaginary directions by substituting $z_j = z_j + \epsilon_{jr} + i \cdot \epsilon_{ji}$ for all j, and grab the coefficients of the ϵ_{jr} and ϵ_{ji}. These coefficients are collected into a matrix, and the eigenvalues of these matrices determine the stability of the equilibrium points.

In [3], Verhulst details how the eigenvalues of a linear system can be used to determine the stability of a solution. The result we use is this:

Given a dynamical system and an equilibrium, let $\{\lambda_j\}$ be the spectrum of the matrix representing the linearized system. Then the solution is

1. asymptotically stable if $Re(\lambda_j) < 0 \quad \forall j$,
2. a saddle if $Re(\lambda_j)$ are of mixed sign,
3. unstable if $Re(\lambda_j) > 0 \quad \forall j$,
4. unknown if $Re(\lambda_j) = 0$ for some j.

In our case, we essentially have two matrices for the perturbed system, one each for the real and imaginary perturbations. Hence, a solution may be stable in the real direction, and unstable in the imaginary, or any other combination of the possibilities.

Note that the complex conjugate operator appearing in the Monks equations is a horribly non-analytic operator. This means that is cannot be represented with a polynomial; Bertini cannot solve the system. Fortunately, because we are only interested in the real solutions to the system, we simply drop the complex conjugate, and assert from here on that z_j is real when we are talking about a solution. For the purposes of stability analysis, however, we use the original equations with the conjugate, and assert z_j real after making the perturbation substitution. Doing so, we get the following matrices for the stability:

$$M_{real} = \begin{bmatrix} \mu_0 - \gamma(z_0^2 + S) & z_2 - 4\gamma z_0 z_1 & z_1 - 4\gamma z_0 z_2 & -4\gamma z_0 z_3 \\ z_2 - 4\gamma z_0 z_1 & \mu_1 - \gamma(z_1^2 + S) & z_0 + z_3 - 4\gamma z_1 z_2 & z_2 - 4\gamma z_1 z_3 \\ z_1 - 4\gamma z_0 z_2 & z_0 + z_3 - 4\gamma z_1 z_2 & \mu_1 - \gamma(z_2^2 + S) & z_1 - 4\gamma z_2 z_3 \\ -4\gamma z_0 z_3 & z_2 - 4\gamma z_1 z_3 & z_1 - 4\gamma z_2 z_3 & \mu_0 - \gamma(z_3^2 + S) \end{bmatrix}$$

$$M_{imag} = \begin{bmatrix} \mu_0 - \gamma(-z_0^2 + S) & -z_2 & z_1 & 0 \\ -z_2 & \mu_1 - \gamma(-z_1^2 + S) & z_0 - z_3 & z_2 \\ z_1 & z_0 - z_3 & \mu_1 - \gamma(-z_2^2 + S) & z_1 \\ 0 & z_2 & z_1 & \mu_0 - \gamma(-z_3^2 + S) \end{bmatrix}$$

Attempts to determine symbolically the eigenvalues for either of these matrices in terms of the four variables and three parameters failed. However, if we have actual values for the parameters and variables, via perhaps *Bertini*, we can easily compute the eigenvalues and make our stability determination.

Using the same data set as in figure 2, we numerically determine the stability for each real solution. In figure 3, we see the number of unstable, unknown, and stable solutions at each point in parameter space. For low γ, there are few solutions to begin with, and we see two distinct regions containing stable solutions. Most solutions are unstable. As γ increases, most solutions remain unstable, with the region of stability merging into one. For all γ scanned, there appear to be two lines of unknown solutions; there are also solutions of unknown stability on the parameter axes, and near the origin.

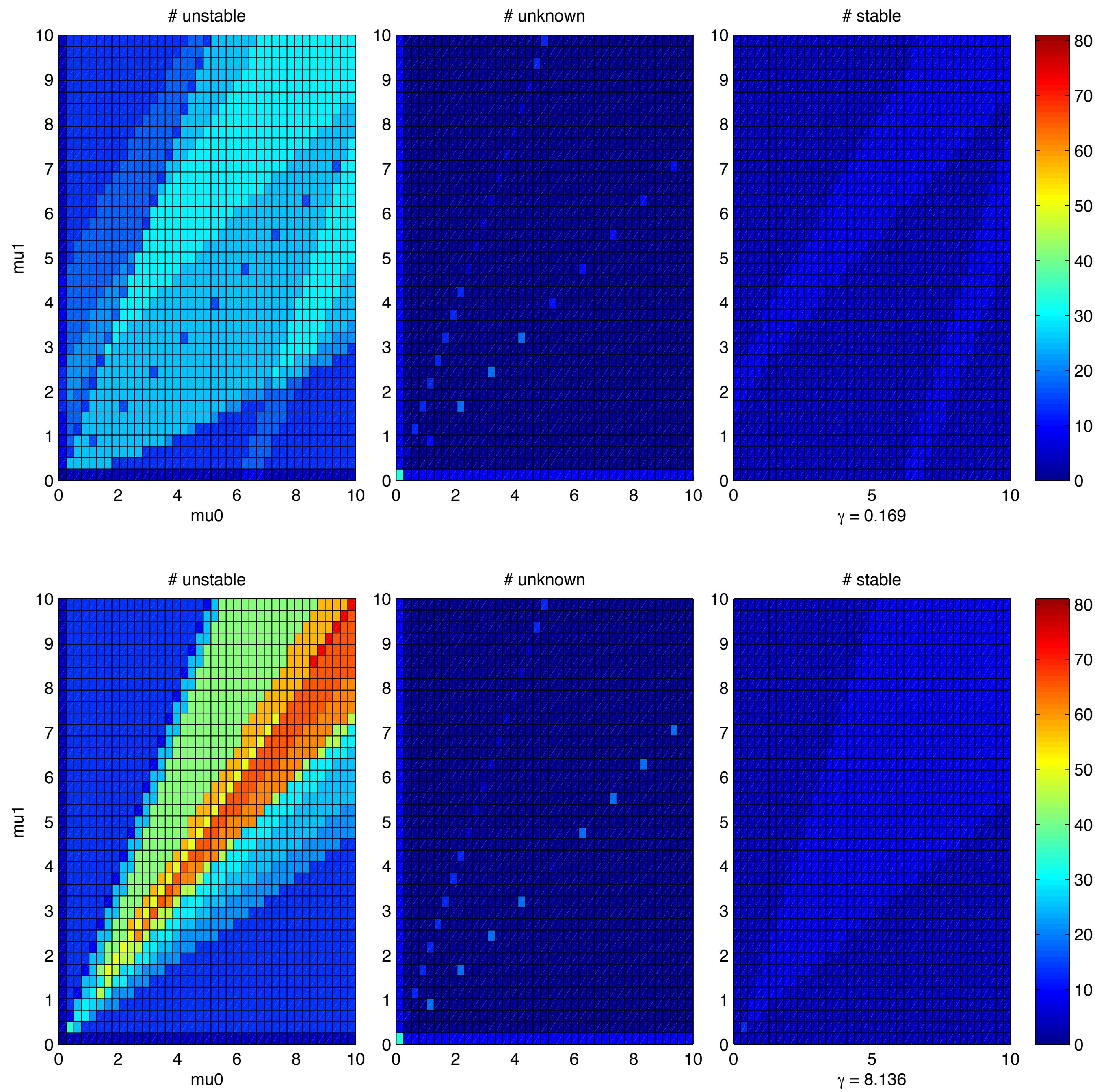

Figure 3: Number of solutions at points in parameter space, left unstable, center unknown due to zero eigenvalue, and right stable. Low γ has two regions of stability, and high γ one. All γ values have two lines of unknown stability. Most solutions are unstable, for all parameter values.

3 More General Resonance Equations

The Monks equations are closely related to another system of resonance equations, again in the Fibonacci style.

Consider waves interacting on an annulus of radius R.

Let the general wave vectors $\bar{k}_i$ be generated from two fundamental waves, $\bar{k}_1$, $\bar{k}_2$, with

$$\bar{k}_1 = (\ell_1\,, 1/R)$$
$$\bar{k}_2 = (\ell_2\,, 2/R)$$

Then the recursive relationship is in the wave vectors as $\bar{k}_{i+2} = \bar{k}_{i+1} + \bar{k}_i$, with the addition taking place component-wise. Furthermore, there is a restriction on the magnitude of the waves, $k_j^2 = \ell_j^2 + \frac{m_j^2}{R^2}$, with m_j as the j^{th} Fibonacci number (this is induced by the recursive definition).

Define the number σ, as

$$\sigma_j = -(k_j^2)^2 + 2Pk_j^2 - 1,$$

and set $S = 2\sum A_j^2$.

Set τ constant, in the range of $[0\,, 3]$, $\gamma \approx 0.1$ to 1, and $\rho \approx 1.1$. The the radius of the disk vary between 1 and two times the largest Fibonacci number under investigation. The variables we seek to find are ℓ_1, ℓ_2, and A_j. We are finally ready to declare the equations in the system. The first two equation are present regardless of how far we go in the sequence:

$$\sum \frac{\partial \sigma_j}{\partial \ell_1} A_j^2 + \tau$$
$$\sum \frac{\partial \sigma_j}{\partial \ell_2} A_j^2 + \tau.$$

We need to go at least as far as $i = 3$ in the recurrence relation in order to write the other equations, which in this particular case are rather simple. Note that asterisk indicates complex conjugation.

$$\begin{aligned}
\sigma_1 A_1 + \tau A_2^* A_3 - \gamma A_1 \left(S - A_1^2\right) &= 0 \\
\sigma_2 A_2 + \tau A_1^* A_3 - \gamma A_2 \left(S - A_2^2\right) &= 0 \\
\sigma_3 A_3 + \tau A_1 A_2 - \gamma A_3 \left(S - A_3^2\right) &= 0
\end{aligned}$$

In the case we go at least four terms in the sequence, the equations get more complicated. In general, the equations are:

$$\begin{aligned}
&\sigma_1 A_1 + \tau A_2^* A_3 && -\gamma A_1 \left(S - A_1^2\right) = 0 \\
&\sigma_2 A_2 + \tau A_1^* A_3 + \tau A_3^* A_5 && -\gamma A_2 \left(S - A_2^2\right) = 0 \\
&\sigma_3 A_3 + \tau A_1 A_2 + \tau A_2^* A_5 + \tau A_5^* A_8 && -\gamma A_3 \left(S - A_3^2\right) = 0 \\
&\sigma_5 A_5 + \tau A_2 A_3 + \tau A_3^* A_8 + \tau A_8^* A_{13} && -\gamma A_5 \left(S - A_5^2\right) = 0 \\
&\vdots
\end{aligned}$$

Using symbolic computation in MATLAB, we automatically generate the *Bertini* input file, and a stability evaluation file. Running the file for a low number of terms ($j = 3$) yields a large number of paths to follow. Upon completion of the step 1 run, for arbitrary complex values of the parameters τ, R, we get some unimportant number of paths. Following these paths, however, we get zero real solutions for all values of the parameters!

Seeing no real solutions for the variables, we then perform a Numerical Irreducible Decomposition on the system. This reveals that this system, no matter how far we go in the Fibonacci sequence, always has positive dimensional components. Our real solutions must therefore lie on these components. This is unfortunate, because so far, there are no implemented speedy methods for finding the real solutions embedded on a positive dimensional complex component.

Our conclusion for this system, therefore, is that while real solutions may exist, we are unable to detect them using the current program. This may lead to further research, in implementing a method for solution. The biggest disappointment perhaps is that there are no pretty pictures in this section of this paper.

References

[1] Monks, Kenneth. Four Waves with a Fibonacci-Like Resonance and Their Symmetries. *Dynamics at the Horsetooth* 1, 1-11, 2009.

[2] Eves, Howard. Great Moments in Mathematics Before 1650. Mathematical Association of America, 1983

[3] Verhults, Ferdinand. Nonlinear Differential Equations and Dynamical Systems. Second Edition. Springer, 2000.