3D printing mathematical surfaces

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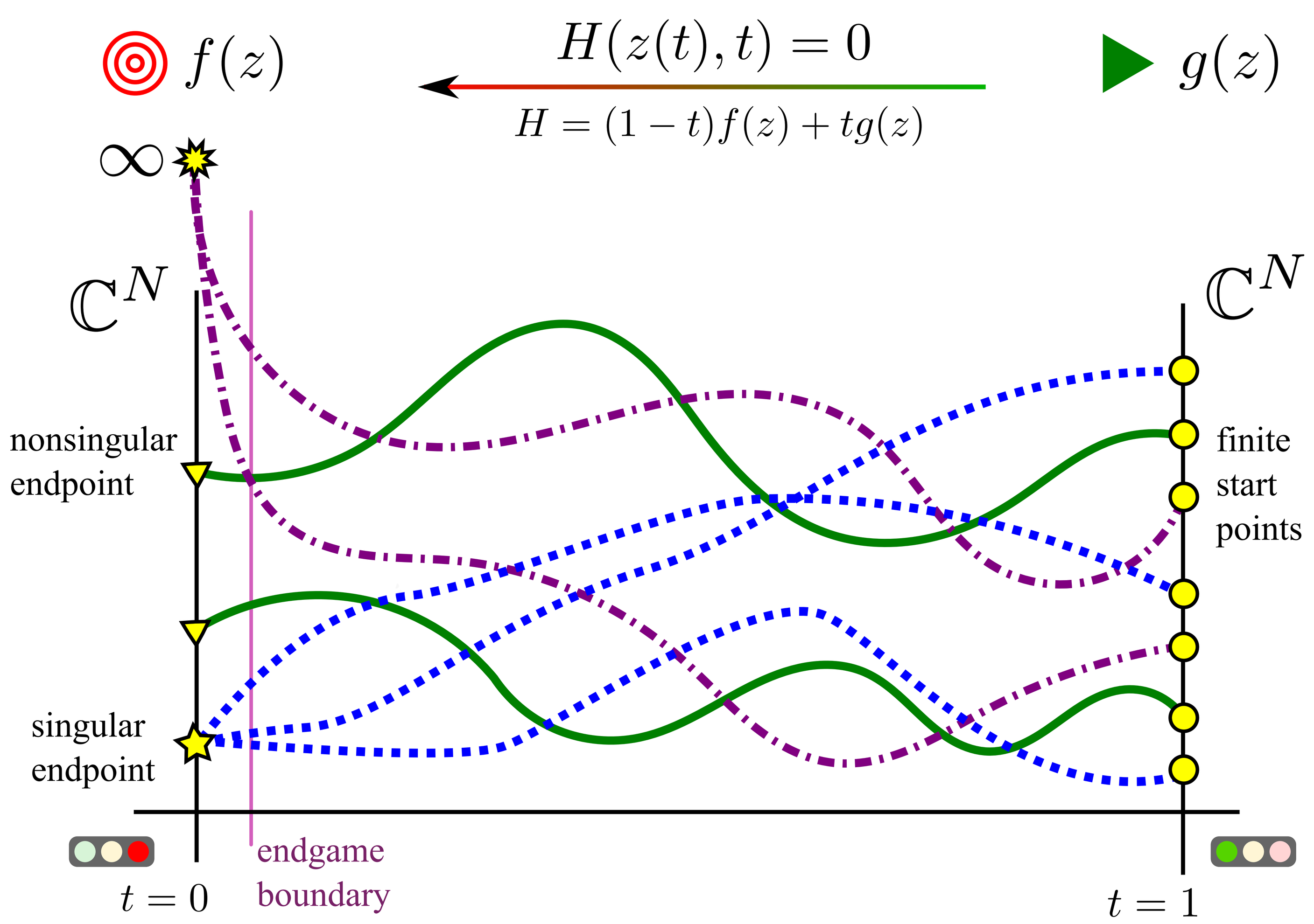
Project goal

To write a computer program which prints polynomials, performing a real numerical cellular decomposition[2,6] of algebraic surfaces using *homotopy continuation*.

Homotopy continuation

Computer programs Bertini and Bertini_real implement *homotopy continuation* to numerically solve $f(z) = 0$, for $\mathbb{C}$ (complex) and $\mathbb{R}$ (real) polynomial systems.



To solve a particular polynomial system $f(z) = 0$,

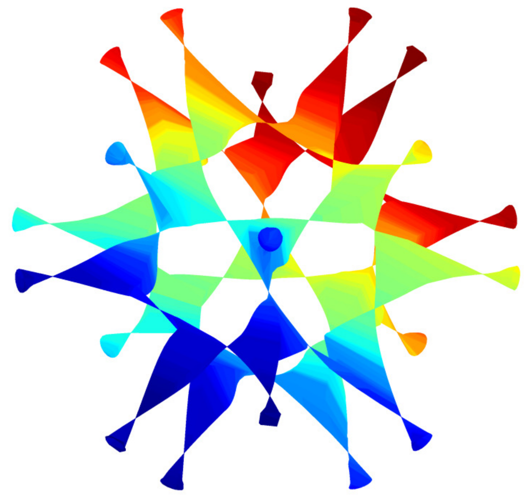
1. analyze the structure of $f(z)$, form a similar system $g(z)$,
2. numerically deform from $g(z)$ to $f(z)$ through time t using a continuous homotopy $H(z(t), t) = 0$.

3D printing polynomials with Bertini_real

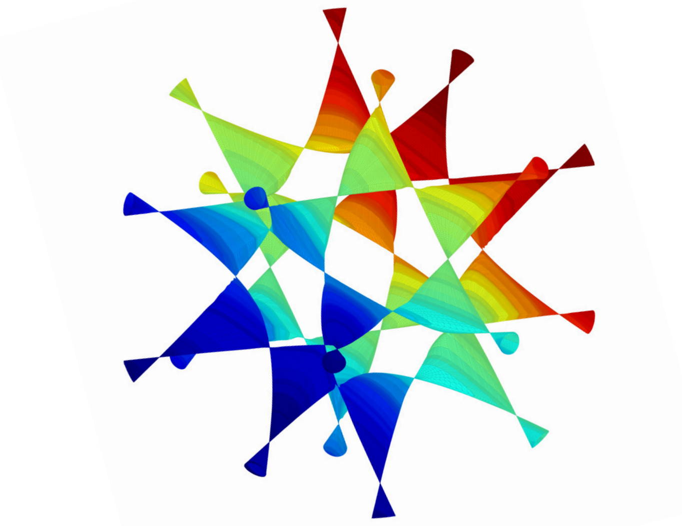
1. Run Bertini[1]
Compute $\mathbb{C}$ witness points

```
***** Witness Set Decomposition *****
| dimension | components | classified
-------------------------------------
|     2     |     1      |     6     |
-------------------------------------
*****  Decomposition by Degree  *****
Dimension 2: 1 classified component
-------------------------------------
degree 6: 1 component
```

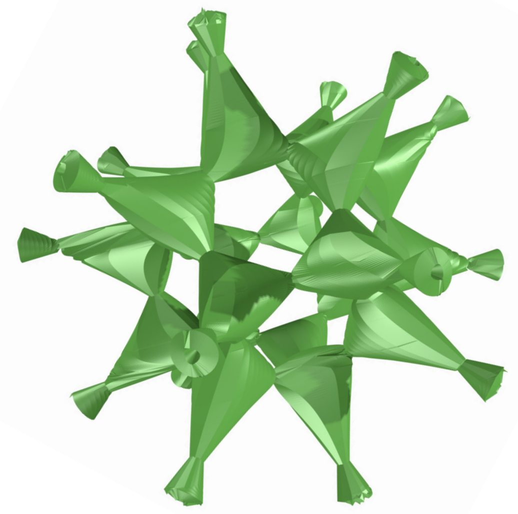
2. Run Bertini_real[3]
Solve many $\mathbb{R}$ polynomial systems



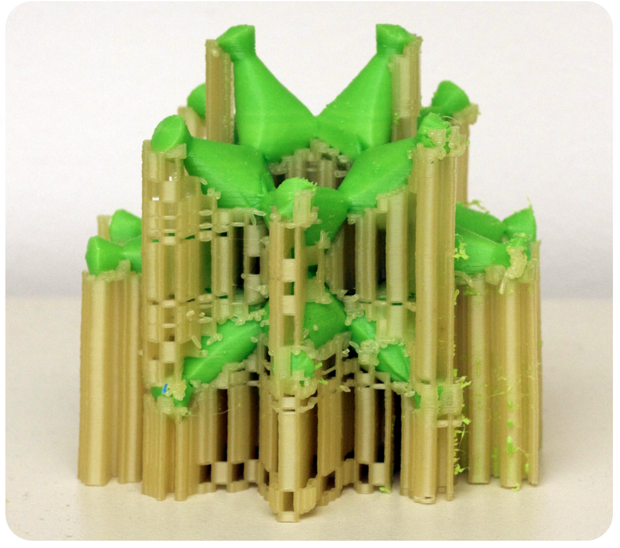
3. Refine
Smooth blocky regions face-by-face



4. Post-process
Solidify, save as .stl



5. Print
Ninjaflex, PVA support



6. Remove support
Warm water bath

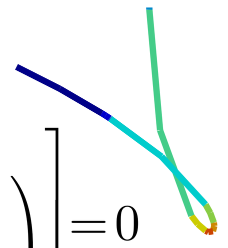
$$4(\phi^2x^2-y^2)(\phi^2y^2-z^2)(\phi^2z^2-x^2)-(1+2\phi)(x^2+y^2+z^2-w^2)^2w^2 = 0, \quad w = 1$$

The Barth Sextic[7]

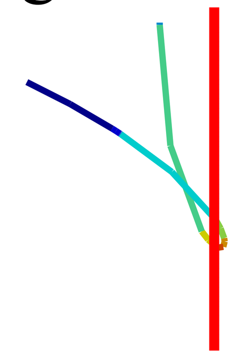
Surface decomposition

1. Decompose critical curve

satisfies $\begin{bmatrix} f(x) \\ \det\left(\begin{bmatrix} Jf(x) \\ \pi_1 \\ \pi_2 \end{bmatrix}\right) \end{bmatrix} = 0$

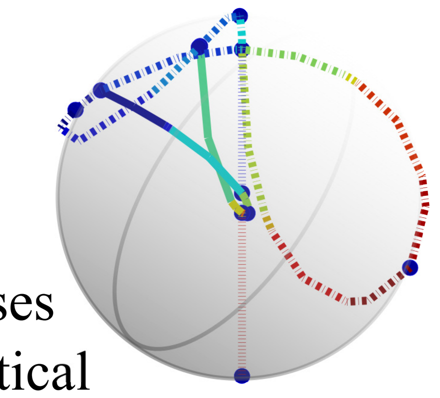


2. Decompose singular curves



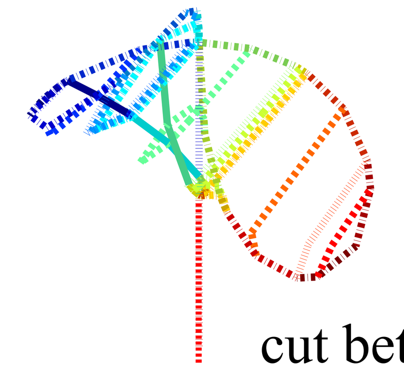
track using isosingular deflation[4]

3. Intersect with sphere



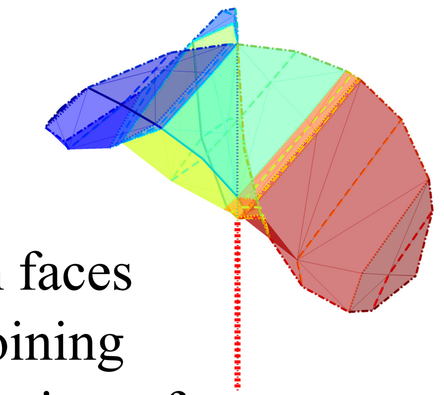
encloses all critical points

4. Slice



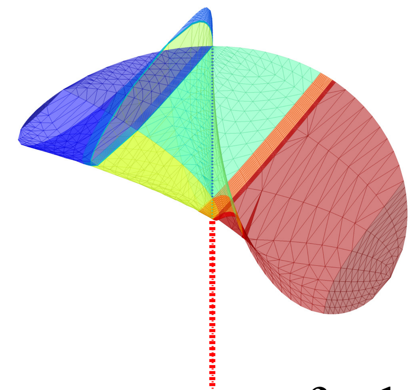
cut between and at each critical point

5. Connect the dots



form faces by joining midpoints of edges of slices

6. Refine



for beauty or science

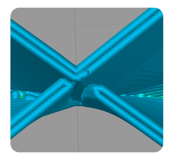
$$x^2 = y^2 z$$

Whitney's Umbrella[5]

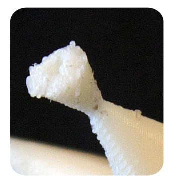
Challenges and future work

Self-crossings and singularities pose a tough problem for 3D printing:

- Solidification is thinner at singularities, leaving gaps in printed surface.



- Folds and cusps have sharp corners. Point singularities have no extent.

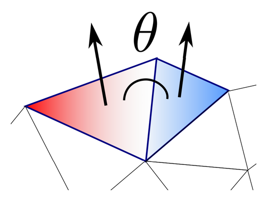


Bertini_real needs optimization:

- Symbolic differentiation for systems with many variables is a major bottleneck.
- Parallelism needs to extend to multiple simultaneous curve decompositions.

Goals for the future:

- Adaptive refinement of high-dimensional surfaces using adjacent angles.



- Decompose and print algebraic 3-folds, or solid regions of space.

Wish list:

- Refrigerated printer enclosure for fast freezing of extruded plastic.
- Full-color printed surfaces for visualizing functions, or a fourth dimension.
- Use Bertini_real to solve, model, and print solutions to engineering problems, e.g. acoustic mirrors.

References and acknowledgements

[1] Bates, Hauenstein, Sommese, Wampler. "Numerically solving polynomial systems with Bertini". SIAM. 2013.

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[3] Brake, Bates, Hao, Hauenstein, Sommese, Wampler. "Bertini_real: software for real algebraic sets". Submitted.

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[5] Hauser. "Algebraic surface gallery". Accessed March 2015. `http://homepage.univie.ac.at/herwig.hauser/bildergalerie/gallery.html`

[6] Lu, Bates, Sommese, Wampler. "Finding all real points of a complex curve". *Contemporary Mathematics* 448:183. 2007.

[7] Weisstein, Eric W. "Barth sextic." From MathWorld – A Wolfram Web Resource. Accessed November 2014. `http://mathworld.wolfram.com/BarthSextic.html`

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