

Applications of computational algebraic geometry

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UWEC Meals with Honors



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AND

Outline

- 1 Polynomials rising
- 2 Computational algebraic geometry
- 3 Specifically Numerical AG
- 4 Two of my projects
- 5 Takeaways

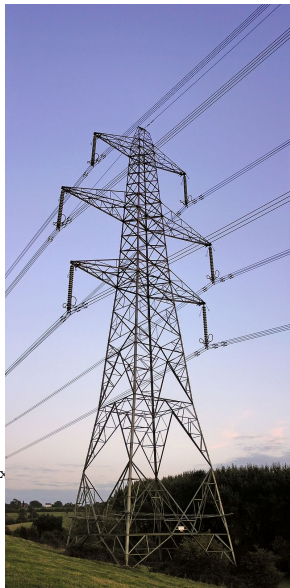
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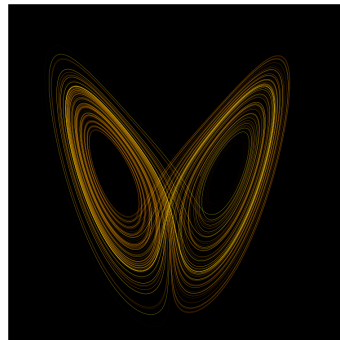
Polynomials are everywhere!



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Polynomials are in ... Networks - Smartgrid



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- Connected components modeled using graph or network theory
- Analysed using polynomial systems
Parameters include
 - ▶ wind speed
 - ▶ solar irradiance
- Compute stability and resilience

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Polynomials are in ... Biology - Synchronization



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Video: <https://www.youtube.com/watch?v=W1TMZASCR-I>

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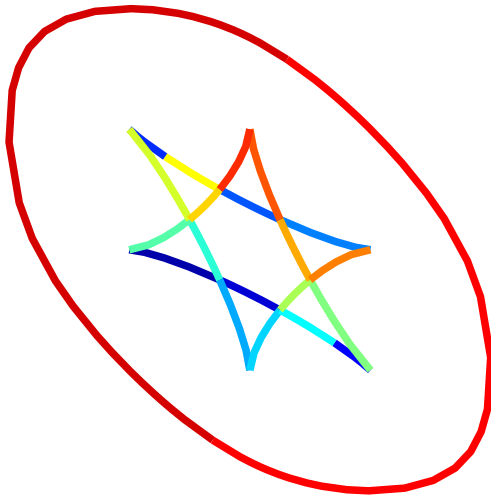
An example determinantal equation for Kuramoto model:

0 =

$$\begin{aligned} &16b_1^8b_2^4 + 64b_1^7b_2^5 + 96b_1^6b_2^6 + 64b_1^5b_2^7 + \\ &16b_1^4b_2^8 + 64b_1^10 + 320b_1^9b_2 + 584b_1^8b_2^2 + \\ &416b_1^7b_2^3 - 128b_1^6b_2^4 - 496b_1^5b_2^5 - 128b_1^4b_2^6 + \\ &416b_1^3b_2^7 + 584b_1^2b_2^8 + 320b_1b_2^9 + 64b_2^{10} - 399b_1^8 - 1596b_1^7b_2 - \\ &934b_1^6b_2^2 + 2784b_1^5b_2^3 + 4643b_1^4b_2^4 + 2784b_1^3b_2^5 - 934b_1^2b_2^6 - \\ &1596b_1b_2^7 - 399b_2^8 + 840b_1^6 + 2520b_1^5b_2 - 772b_1^4b_2^2 - 5744b_1^3b_2^3 - \\ &772b_1^2b_2^4 + 2520b_1b_2^5 + 840b_2^6 - 766b_1^4 - 1532b_1^3b_2 - 2298b_1^2b_2^2 - \\ &1532b_1b_2^3 - 766b_2^4 + 288b_1^2 + 288b_1b_2 + 288b_2^2 - 27 \end{aligned}$$

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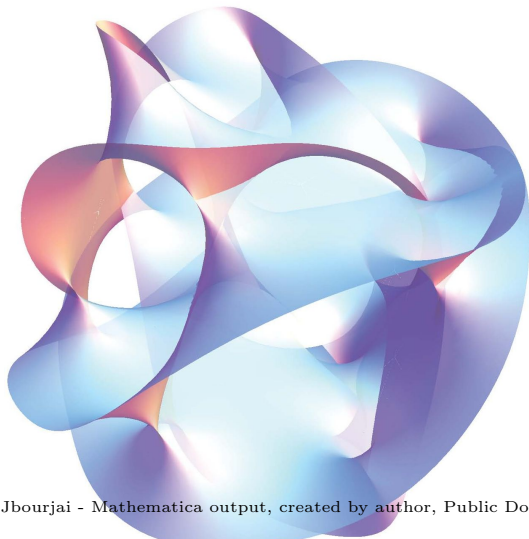


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Polynomials are in ... Theoretical physics

Calabi-Yau manifolds



In *Quantization of Non-critical Bosonic Open String Theory*, e.g., solve polynomials to prove consistency of theory. Used *Gröbner bases*

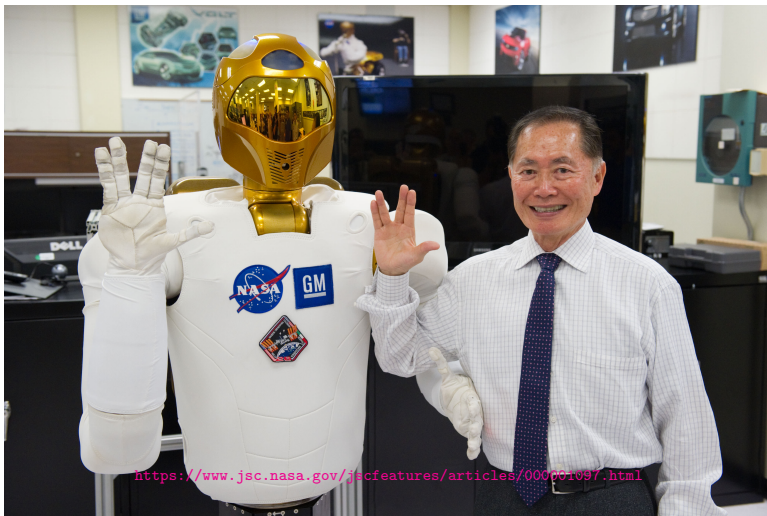
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Polynomials are in ... Robotics

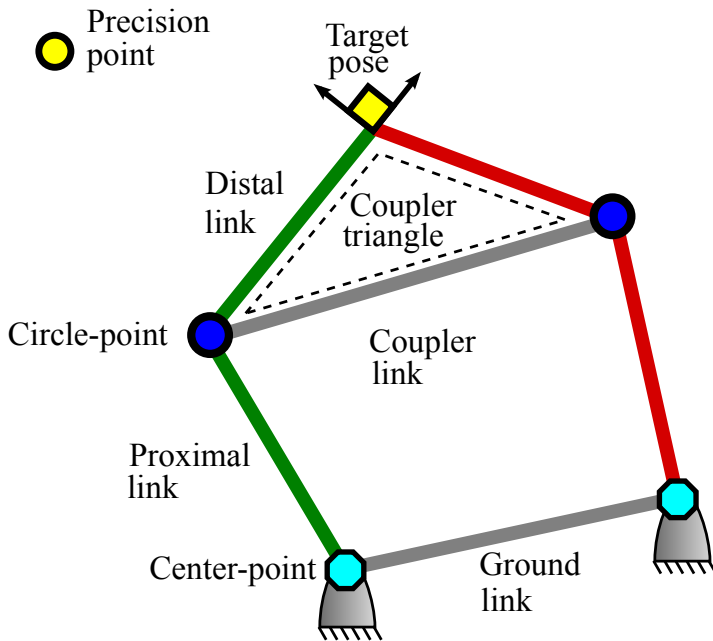


Robonaut, designed by solving polynomial systems

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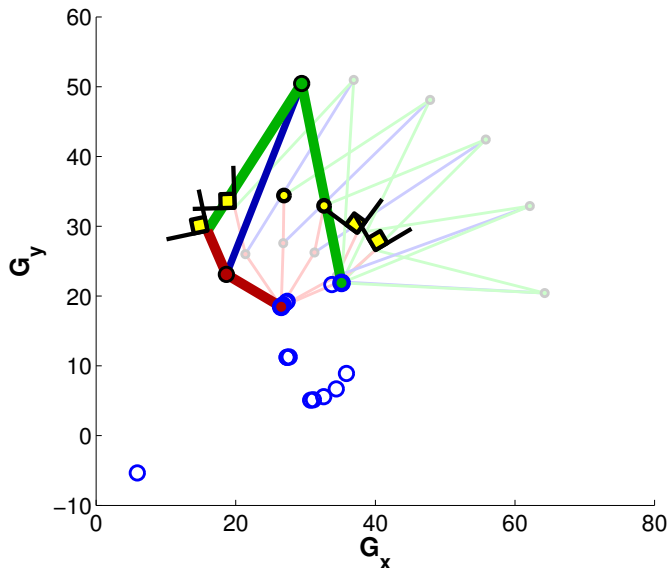
Design a four-bar



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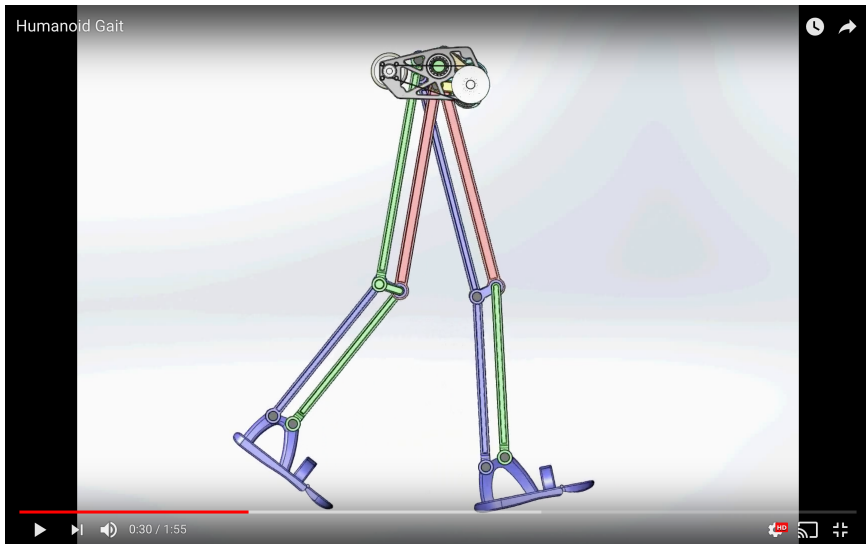
Design a four-bar



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You can compute a walking leg

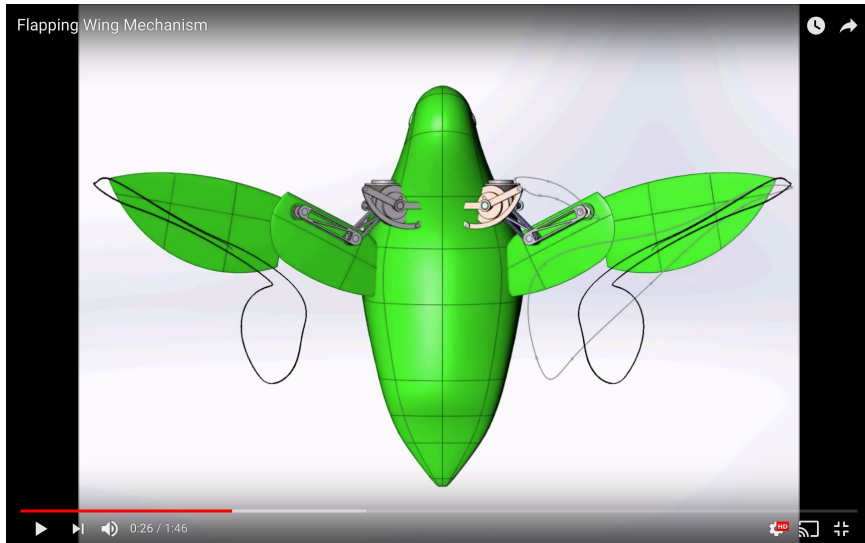


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<https://www.youtube.com/watch?v=PKavSYZcdGU>

You can compute a bird wing



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<https://www.youtube.com/watch?v=7aXmze9Ynis&feature=youtu.be>

Polynomials are in ... Computer vision

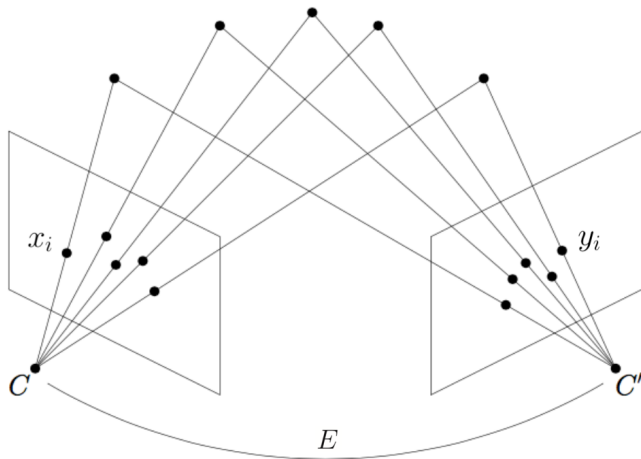


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Polynomials are in ... Computer vision



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Polynomials are in ... Computer vision

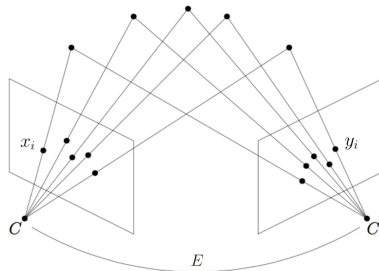


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5-Point Problem



- The 5-point problem is a problem with 10 solutions.
- Two cameras have data on 5 points related to an object in space.

Goal: Find the relationship between the data from the two cameras
 $\Rightarrow$ be able to determine the location of the 3-D object.

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5-Point Problem Polynomial System

The polynomial system for the 5-point problem is:

$$\begin{bmatrix} 2EE'E - \text{trace}(EE')E \\ y'_i E x_i \end{bmatrix} = 0$$

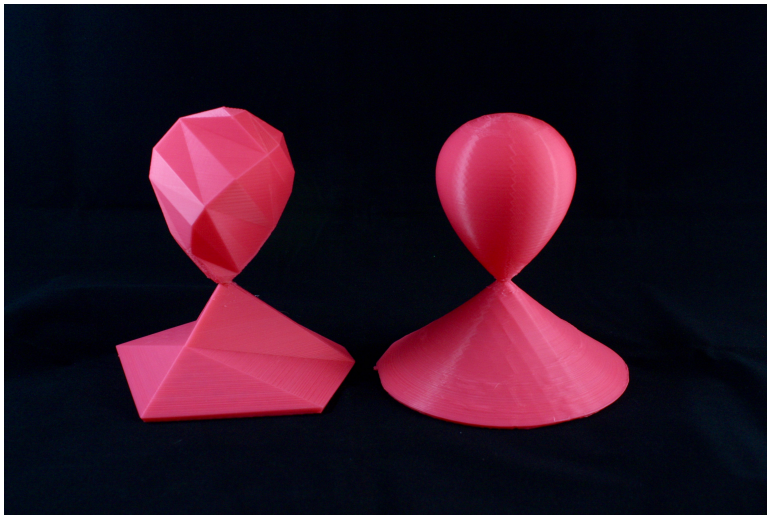
- E is a 3×3 matrix defined up to scale (in projective space).
 $\Rightarrow$ Emits 9 equations.
- The parameters x_i and y_i are points in the plane
 $\Rightarrow$ 5 total pairs.
- This is overdetermined! (14 equations with 8 variables).
 $\Rightarrow$ “Should” be impossible, but magically isn’t, yay!

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Polynomials are in ... 3D printing



$$x^2 + y^2 + z^3 = z^2$$

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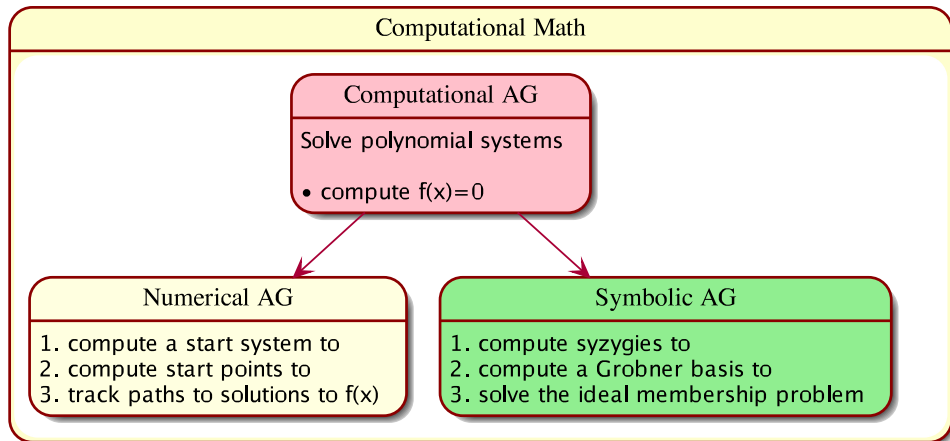
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Overview - branching



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```
> ring r=32003,(a,b,c,d), lp;  
> ideal I = cyclic(4);  
> ideal sI = std(I);  
> sI;  
sI[1]=c2d6-c2d2-d4+1  
sI[2]=c3d2+c2d3-c-d  
sI[3]=bd4-b+d5-d  
sI[4]=bc-bd5+c2d4+cd-d6-d2  
sI[5]=b2+2bd+d2  
sI[6]=a+b+c+d  
>
```

- Given polynomials, got back other polynomials
- New polynomials form a *basis*, in that any other polynomial is uniquely decomposable
- Main algorithm is *Buchberger's algorithm*, and it computes a *Gröbner basis*

Software packages:

-  SINGULAR
- Macaulay2
- Polymake
- CoCoA
- GAP Gnu Algebra Program (mainly algebra)
- MAGMA (mainly algebra)
- IMO you kinda have to be an algebraist or algebraic geometer to understand how to use these packages
- Guaranteed correct answers
- Gives you back more polynomials, with exact coefficients
- Many have numerical methods implemented, or links to NAG software

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Bertini1 output is points

Tracking path 0 of 24
Tracking path 20 of 24

Finite Solution Summary

NOTE: nonsingular vs singular is based on condition number and identical endpoints

	Number of real solns	Number of non-real solns	Total
Non-singular	0	0	0
Singular	4	4	8
Total	4	4	8

Finite Multiplicity Summary

Multiplicity	Number of real solns	Number of non-real solns
2	0	4
3	4	0

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Bertini1 output is points

12

```
-0.100000000000000001985254058589094e1 0.228010770266136771731388218135e-14  
0.9999999999999991792266140901537e0 0.770118487507406471873746848380e-16  
0.100000000000000001797295503188021e1 -0.111744575256855163771961008905e-14  
-0.10000000000000000318524384870179e1 -0.958827658531361352532710580819e-15  
  
-0.9999999999999994220276540519098e0 -0.383563170932190311006809397822e-15  
0.9999999999999998265213115480175e0 0.166169862805964949383533078344e-15  
0.1000000000000000044632726565051e1 0.804768507174298789638260327843e-15  
-0.10000000000000000381902880518924e1 -0.116412178945711947812980207799e-14  
  
-0.9999999999999996074135200751701e0 -0.209717066263946547639337710644e-14  
0.100000000000000001224106953070615e1 0.989763273931492574031991105489e-15  
0.999999999999999126758994344381e0 -0.982279165307545323608238531998e-15  
-0.9999999999999997114579184528211e0 0.737356293048803522475076234758e-15
```

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Look, coordinates!

Software packages:

- $\mathbb{B}^2$ Bertini
- Hom4PS
- PHCPack
- NAG4M2
- Approximate solutions
- No exact coefficients
- Can prove have correct solutions – sometimes
- Faster, sometimes

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There's not a clear winner

Symbolic

- Guaranteed correct solutions / results
- Great when:
 - ▶ you are willing to wait a while if necessary
 - ▶ you need exact numbers or relations
 - ▶ you have small systems



Numeric

- Probabilistically correct solutions
- Great when:
 - ▶ you need super-quick runtimes
 - ▶ you are willing to accept errors
 - ▶ you have a large^a system

^aI don't know what this means

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Homotopy continuation – an example

Let's solve this two-variable *target system*:

$$f(x, y) = \begin{bmatrix} x^{10}y^3 - \sqrt{3}x^2 + 2xy - 4 \\ \frac{1}{2}x^4y^4 - x^3(1 + x^4) + 1 \end{bmatrix}$$

To do so, let's instead solve this easier *start system*:

$$g(x, y) = \begin{bmatrix} x^{13} - 1 \\ y^8 - 1 \end{bmatrix}$$

Then we *continue* g into f using this *homotopy*:

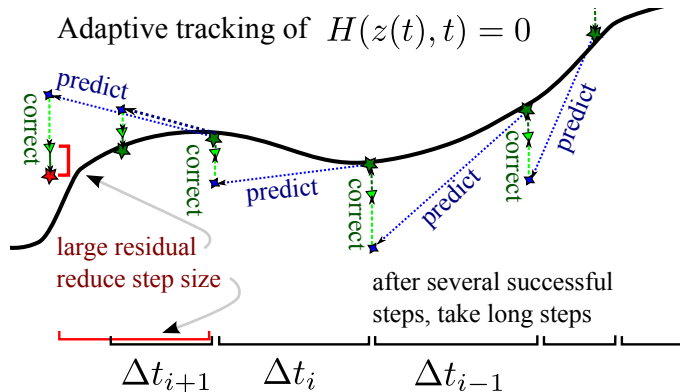
$$H(x, y, t) = (1 - t)f + tg = 0$$

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University of Wisconsin
Eau Claire

Predictor-corrector tracking



Solve the *Davidenko Differential Equation*

$$\frac{\partial H}{\partial z} \frac{dz}{dt} + \frac{\partial H}{\partial t} = 0 \quad \Rightarrow \quad \frac{dz}{dt} = - [JH(z, t)^{-1}] \frac{\partial H(z, t)}{\partial t}$$

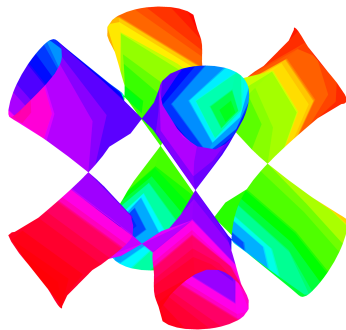
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Why does NAG work?

- Bertini's theorem – a generalization of the
fundamental theorem of algebra.

The FToA tells us that
 $f(z)$ is a univariate polynomial
of degree d ,
 $\Rightarrow f$ has d roots over $\mathbb{C}$, counting
multiplicity



$\Rightarrow$ This surface intersects a
random complex four times

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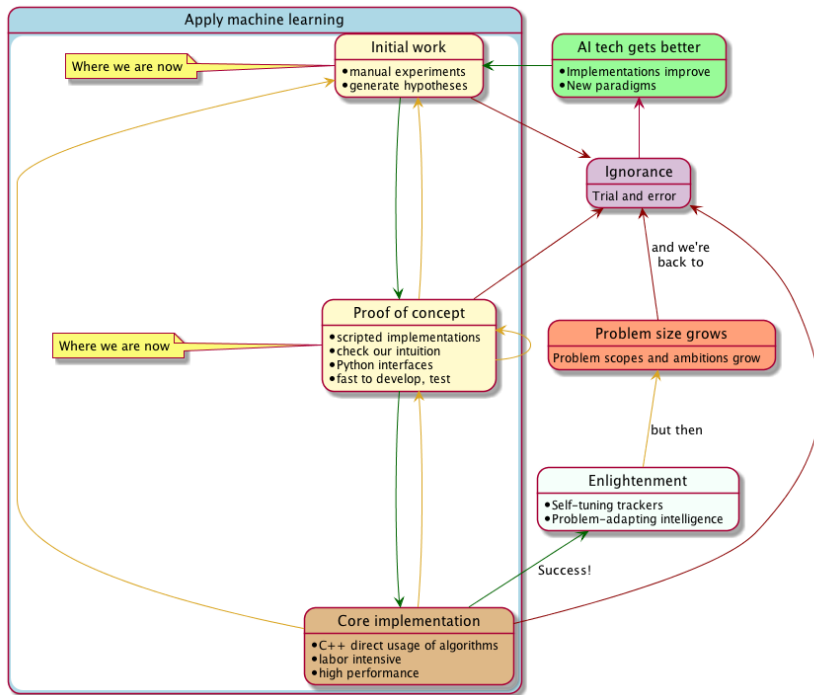


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Herwig's gallery is wonderful

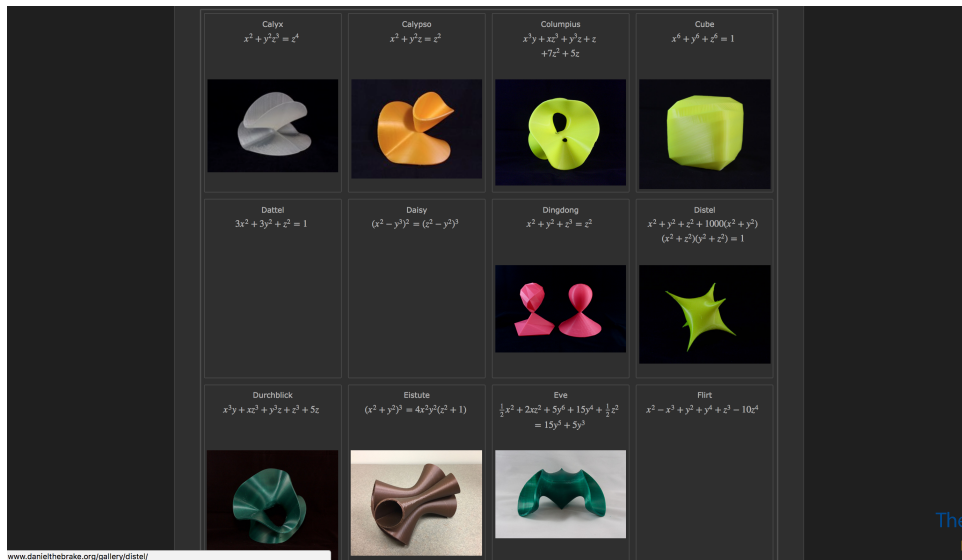


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<http://homepage.univie.ac.at/herwig.hauser/bildergalerie/gallery.html>

My gallery is incomplete



www.danielethebrake.org/gallery/distel/

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<https://danibrake.org/gallery>

Surface decomposition is implicit function theorem

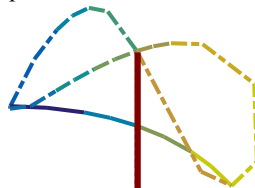
1. Decompose
critical curve



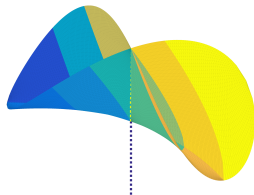
2. Decompose
singular curves



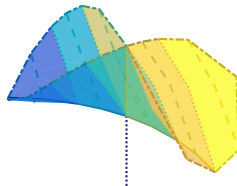
3. Intersect with
sphere



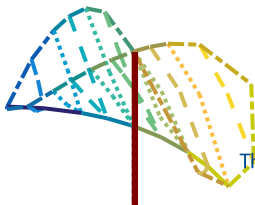
6. Refine



5. Connect the dots



4. Slice



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You have a system? We will solve it



PublicDomain, <https://commons.wikimedia.org/w/index.php?curid=656166>

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Computing $f(x) = 0$ is not a solved problem

Largest systems solved to date:

- The kinematic design systems from Mike McCarthy's group
 - ▶ Bipedal gait machine - 705,432 paths - kilodegree
 - ▶ Flapping bird wing - 300 million paths - megadegree
 - ▶ coming soon? - > 1 billion paths - gigadegree

Next steps to success?

- Ever-better parallelism – fully utilize large-scale clusters
- Specialized and optimized routines for particular types of systems
- Libraries of pre-solved systems - hosted on the cloud

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Join our welcoming community



Jon and Andrew



Maggie and Charles

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Thank you for your kind attention!



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