Numerically decomposing algebraic surfaces with an infinite number of singularities

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with
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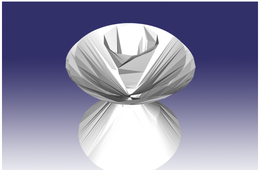
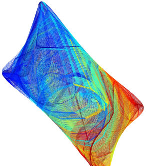




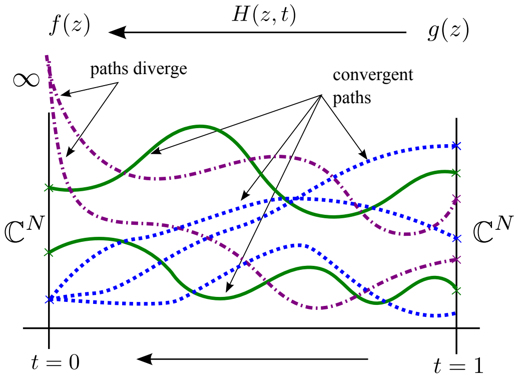
Polynomial Systems

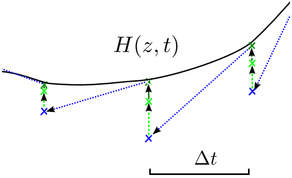
Polynomials are everywhere!

- chemical reaction networks
- kinematics
- dynamical systems
- optimization
- many more

How to solve a system $f(z)$? That's **hard**!

Homotopy Methods





Start Systems

There are lots of ways to make start system g.

- Roots of unity - $g_i = (z_i - 1)_i^d$
- Total degree - $g_i = \prod^{d_i} \mathcal{L}(\vec{x})$
- Multi-homogeneous - $g_i = \prod \mathcal{L}(\vec{x}_\sigma) \cdot \prod \mathcal{L}(\vec{x}_\delta) \cdot \ldots$
- Polyhedral - g_i determined by the Newton Polytope

The canonical linear homotopy:

$$H(z,t) = t \cdot f(z) + \gamma(1-t) \cdot g(z)$$

Witness Sets

Witness set - Numerical Algebraic Geometry's representation of positive dimensional components C.

Consists of three things:

- points x – generic, nonsingular. $\# = \text{degree}(C)$.
- linears L – generic. $\# = \dim(C)$.
- system f – possibly randomized, Jf is full rank on each x.

Positive dimensional $\mathbb{C}$ components are nice - degree and structure are same almost everywhere.

$\mathbb{R}$ not so much...

Real Components

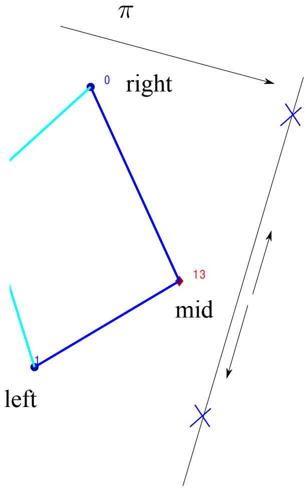
Bertini_real – Numerical Cellular Decomposition

Setup

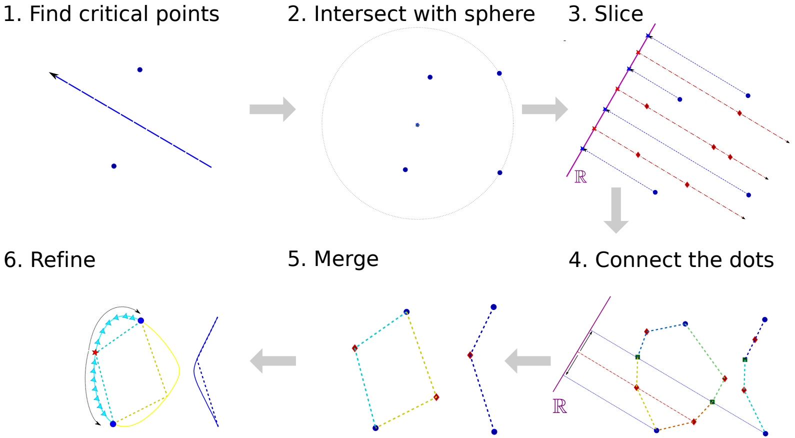
- Let f be a polynomial system with $\mathbb{R}$ coefficients, and $f : \mathbb{C}^N \to \mathbb{C}^n$.
- Let $V(f)$ be the variety of f.
- Consider $C \subseteq V(f)$ be a component of dimension k.
- If f is overdetermined, replace f by a randomized version of itself.

Our objective is to decompose the real part of C; i.e., $C \cap \mathbb{R}^N$

Curve Cell

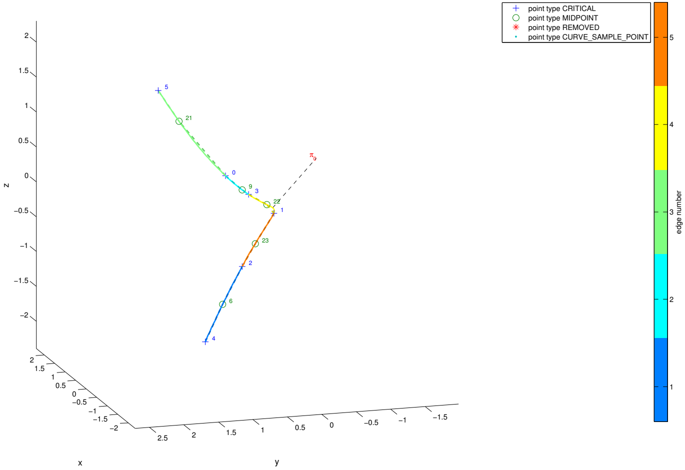


Curves



[Lu,Bates,Sommese,Wampler,2006]

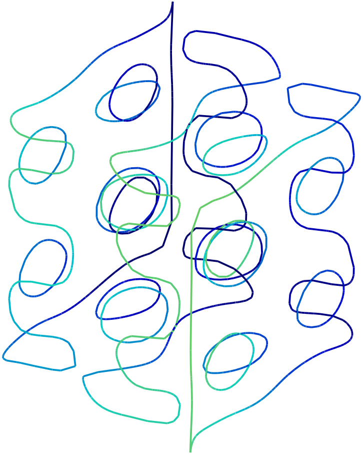
Curves



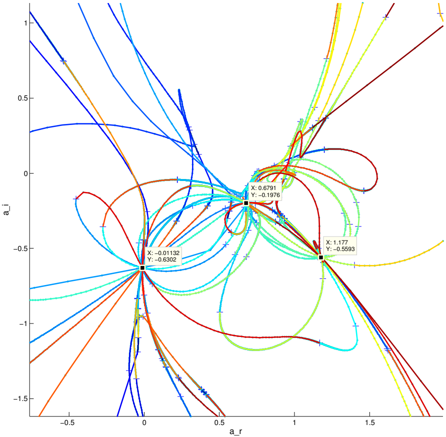
twisted cubic

$$f(x,y,z) = \begin{bmatrix} y - x^2 \\ z - x^3 \\ y^2 - xz \end{bmatrix}$$

Curves



Curves



Burmester 3-3 curve
dimension 14. In 2d projection, of degree 128.

Critical points of curves

Computing critical points of curves is easy.

Since f defines a dimension-one component, the working witness set comes with one random linear:

$$\begin{bmatrix} f \\ \mathcal{L}_1 \end{bmatrix}$$

Then we use regeneration to solve the system

$$\begin{bmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \end{pmatrix} \\ \text{patch}_v \end{bmatrix}$$

[Hauenstein,Sommese,Wampler, 2009]

Regeneration to find crit

Regeneration uses products of linears to build up a start system for a homotopy. We're solving by homotoping as

$$H(x,v;t) = (1-t)\begin{bmatrix} f(x) \\ v^\intercal \cdot \begin{bmatrix} Jf(x) \\ J\pi_1 \end{bmatrix} \\ \text{patch}_v \end{bmatrix} + t \begin{bmatrix} f(x) \\ M_1(v)\prod_{i=1}^{\delta} L_{1,i}(x) \\ \vdots \\ M_N(v)\prod_{i=1}^{\delta} L_{N,i}(x) \\ \text{patch}_v \end{bmatrix} = 0$$

- δ is the maximum degree any polynomial in f.

We have a new result supporting this computation – a single homotopy of this form will find all isolated solutions in terms of x variables, regardless of the fiber dimension.

Building a start system

- To form the start system and solutions, we move the given random complex $\mathcal{L}(x)$ to each $L_{j,i}$ one at a time, to find the x coordinates:

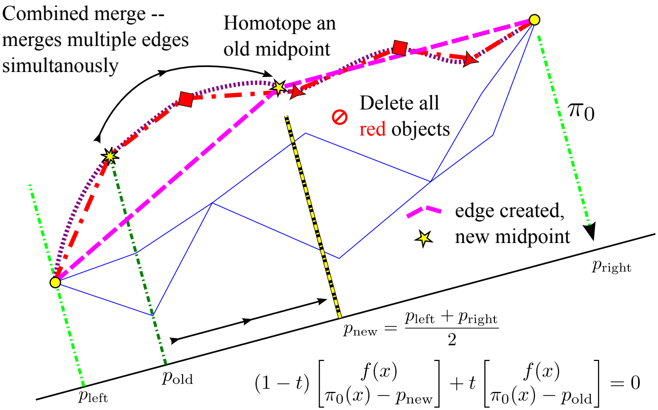
$$H(x;t) = (1-t)\begin{bmatrix} f \\ L_{j,i} \end{bmatrix} + t\begin{bmatrix} f \\ \mathcal{L}_1 \end{bmatrix} = 0$$

- Since only one $L_{j,i}$ vanishes for each start point, the remainder of the start functions must vanish due to $M_j(v)$, which are solved for v start values by matrix inversion:

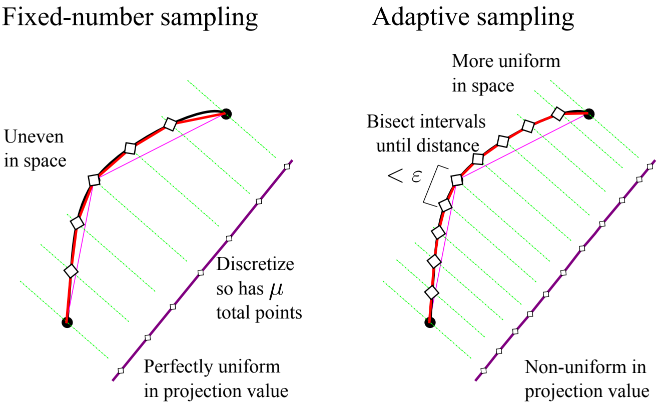
$$v = \begin{bmatrix} M_{1\neq j} \\ \vdots \\ M_{N\neq j} \\ \text{patch}_v \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Then we take the x solution from the top, and v from bottom, and concatenate to form the start point.

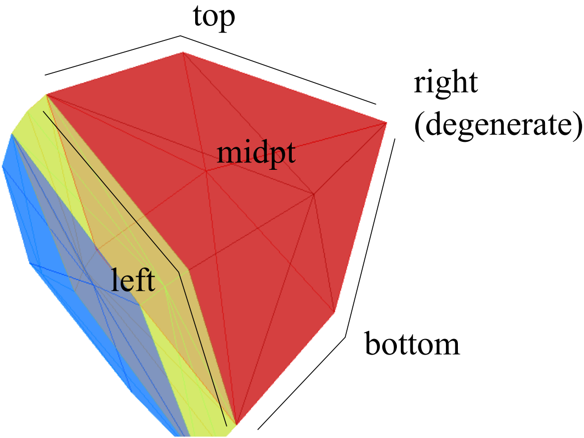
Curve Merging



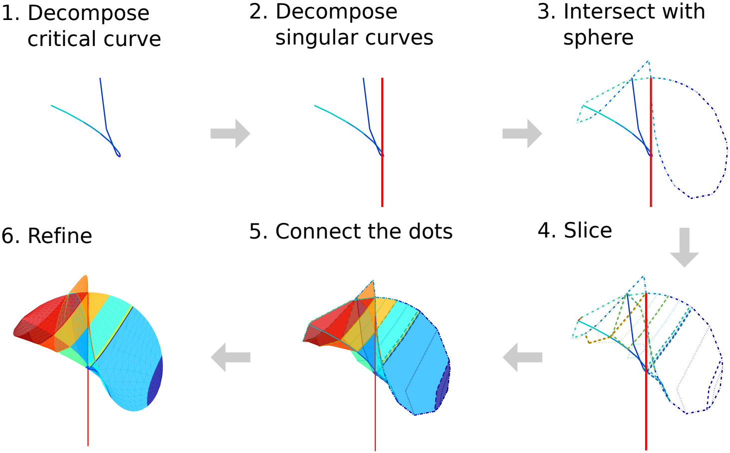
Curve Sampling



Surface Cell

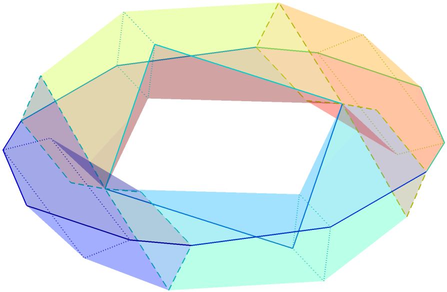


Surface Decomposition



[Besana,Di Rocco,Hauenstein,Sommese,Wampler, 2013]

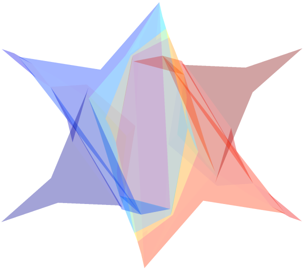
Surface examples



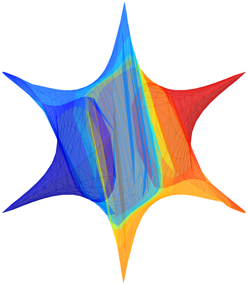
torus

$$f(x,y,z) = (x^2 + y^2 + z^2 + 2^2 - \frac{1}{2}^2)^2 - 16(x^2 + y^2)$$

Surface examples



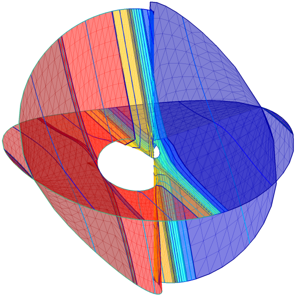
distel, unrefined



distel, refined

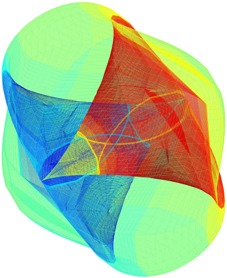
$$f(x,y,z) = x^2 + y^2 + z^2 + 1000(x^2 + y^2)(x^2 + z^2)(y^2 + z^2) - 1$$

Surface examples



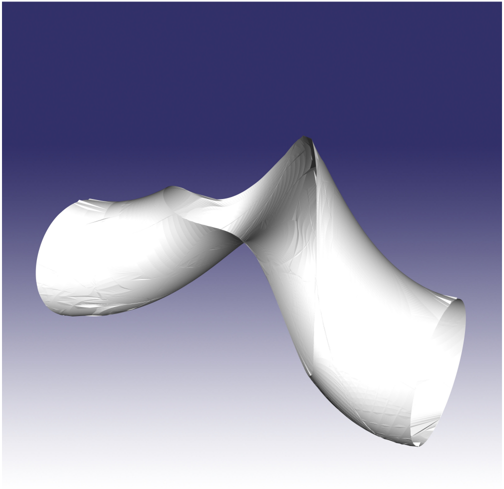
solitude

$$f(x,y,z) =$$
$$x^2yz + xy^2 + y^3 + y^3z - x^2z^2$$



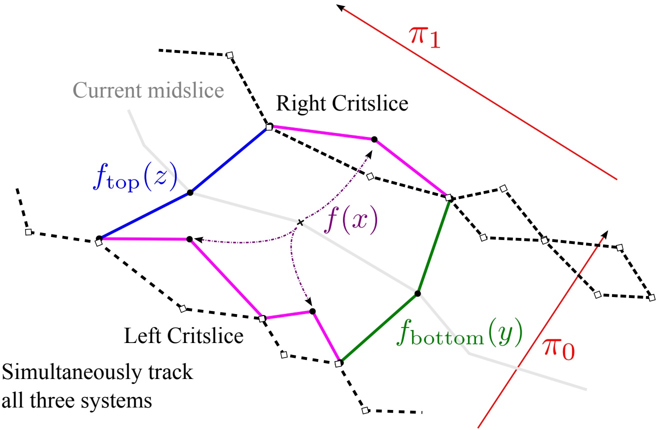
klein

$$f(x,y,z,w) =$$
$$\begin{bmatrix} w^2 + x^2 + y^2 + z^2 - 1 \\ 2wyz - x(y^2 + z^2) \end{bmatrix}$$



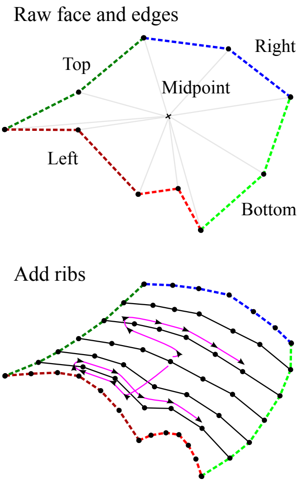
Fivebar mechanism kinematics
Six-dimensional trigonometric system,
passed though an atan2 projection

Midtracking



Simultaneously track
all three systems

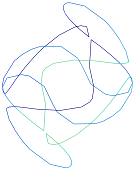
Surface Refinement



Crit points of surfaces

Computing critical points of surfaces is hard.

- Surfaces have *critical curves*.
- Surfaces also have critical points themselves.



Current surface critical method

Using the *determinantal* form of the criticality conditions:

witness set:

$$f_{\text{crit\_curve}} = \begin{bmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \\ J\pi_2 \end{pmatrix} \\ \mathcal{L}_1 \end{bmatrix}$$

$$f_{\text{crit\_crit}} = \begin{bmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \\ J\pi_2 \end{pmatrix} \\ \det \begin{pmatrix} J \begin{pmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \\ J\pi_2 \end{pmatrix} \end{pmatrix} \\ J\pi_1 \end{pmatrix} \\ \text{patch}_v \end{bmatrix}$$

- Determinant operation produces high degree polynomials, contributing to numerical issues.
- Using this formulation for dimension 3 decompositions will be even worse w.r.t. degree and computational complexity.
- Fortunately, we can still use the nullspace method for finding critical points of this curve.

Crit points of crit curve

The actual system Bertini_real currently solves to obtain crit points of crit curve, by regeneration:

$$f_{\text{crit\_crit}} = \begin{bmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \\ J\pi_2 \end{pmatrix} \\ v^\intercal \cdot J \begin{pmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \\ J\pi_2 \end{pmatrix} \\ J\pi_1 \end{pmatrix} \\ \text{patch}_v \end{bmatrix}$$

How to find singular curves

1. Compute witness set for critical curve.
2. Separate singular points from nonsingular.
   - Nonsingular $\to$ critical curve.
   - Singular $\to$ singular curve(s).
3. Separate singular witness points by deflation sequence.
4. Decompose each singular curve.

Deflation

Deflation

- making the untrackable, trackable.
- Removing singularity of points.

Multiple flavors of deflation:

- Nullspace - adds **lots** of variables, but no functions

$$g_i(x, \xi_1, \xi_2, \ldots, \xi_d) = \begin{bmatrix} f(x) \\ Jf(x) \cdot (p_1 + A\xi_1) \\ \vdots \\ Jf(x) \cdot (p_d + A\xi_d) \end{bmatrix}$$

Notation - d is dimension of nullspace of Jf.

[Hauenstein, Wampler '13]

Deflation

- Determinental - adds no variables, but **lots** of functions

$$g_{\text{det}} = \begin{bmatrix} f(x) \\ \det J_{\sigma_1} f(x) \\ \vdots \\ \det J_{\sigma_m} f(x) \end{bmatrix}$$

- σ_i is an index set of an $(N-d+1) \times (N-d+1)$ submatrix.
- $m = \binom{n}{N-d+1} \cdot \binom{n}{N-d+1}$

For our decompositions, we use *isosingular deflation*, and the *determinental* formulation

[Hauenstein, Wampler '13]

Isosingular Deflation

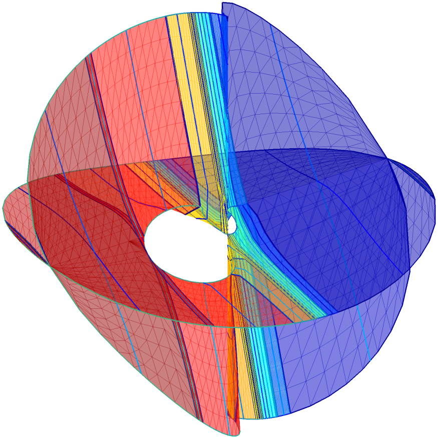
Separate points based on *deflation sequence.*
- dimension of nullspace after r^{th} deflation:

$$\{d_r(f,V)\}_{r=0}^{\infty}$$

Method

1. Compute witness points for critical curve – some may be singular.
2. Deflate surface WRT a singular witness point - get sequence.
3. Test other witness points to see if satisfy deflated system.
4. GOTO 2, for each non-deflated point.
5. Decompose curves, using deflated system.

Example - Solitude



Example - Solitude

Solitude:

$$f(x,y,z) = x^2yz + xy^2 + y^3 + y^3z - x^2z^2$$

$$\mathcal{D}^1 = \begin{bmatrix} x^2yz + xy^2 + y^3 + y^3z - x^2z^2 \\ \\ y^2/4 - (xz^2)/2 + (xyz)/2 \\ (xy)/2 + (x^2z)/4 + (3y^2z)/4 + (3y^2)/4 \\ (x^2y)/4 - (x^2z)/2 + y^3/4 \end{bmatrix}$$

Example - Solitude

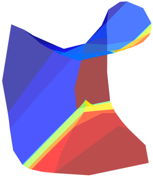
$$
\mathcal{D}^2 = \left[\begin{array}{c}
x^2yz + xy^2 + y^3 + y^3z - x^2z^2 \\
\\
y^2/4 - (xz^2)/2 + (xyz)/2 \\
(xy)/2 + (x^2z)/4 + (3y^2z)/4 + (3y^2)/4 \\
(x^2y)/4 - (x^2z)/2 + y^3/4 \\
\\
(y^2z^2)/8 - (x^2z^3)/24 + (y^2z^3)/8 - (y^3z^2)/8 - (y^3z)/8 + y^3/24 + (xy^2z)/24 + (x^2yz^2)/24 \\
(x^2z^3)/12 + (y^3z^2)/24 + (xy^3)/24 - (y^4z)/24 - (xy^2z)/12 - (x^2yz^2)/8 + (x^2y^2z)/24 \\
(x^2y^2)/24 + (xy^3)/8 - y^4/24 - (xy^2z)/4 - (x^2yz)/12 + (xy^3z)/12 - (xy^2z^2)/4 \\
-(x(x^2z^2 - 3y^2z^2 + 2xz^2 + 6yz^2 - 3y^2z + 6yz^3 + y^2 + xyz))/24 \\
(x^3z^2)/24 - (x^2y^2)/48 + y^4/48 + (x^2yz)/12 + (xy^3z)/12 - (xy^2z^2)/8 \\
(xy^3)/12 - (x^2y^2)/16 + (x^3z)/12 + (x^4z)/48 + (y^4z)/16 + y^4/16 + (x^2yz)/4 + (x^2yz^2)/4 \\
(x^3z^2)/24 - (x^2y^2)/16 - (xy^3)/8 + y^4/16 + (xy^2z)/4 + (x^2yz)/6 + (xy^2z^2)/8 \\
-(x(x^2y^2 + 2x^2z^2 + xy^2 - 2y^3z + y^4 - 2x^2yz))/24 \\
-(y(4x^2y^2 + 6x^2y + 4x^3 + x^4 + 3y^4))/48 \\
(y^2z^2)/12 - (x^2z^2)/36 - (xz^2)/36 - (yz^2)/12 + (y^2z)/12 - (yz^3)/12 - y^2/36 - (xyz)/36 \\
(x^2z^2)/24 - (y^2z^2)/24 - (xy^2)/36 + (y^3z)/24 - (x^2yz)/72 + (xyz)/18 \\
(x^2z)/18 - (x^2y)/72 - (xy^2)/12 + (x^3z)/72 + y^3/24 + (xyz^2)/6 - (xy^2z)/24 + (xyz)/6 \\
(x^2z^2)/24 - (y^2z^2)/24 - (xy^2)/36 + (y^3z)/24 - (x^2yz)/72 + (xyz)/18 \\
-(x^2(y^2 - 3yz + 3z^2))/36 \\
-(xy(2x - 6yz + x^2 + 3y^2))/72 \\
(x^2z)/18 - (x^2y)/72 - (xy^2)/12 + (x^3z)/72 + y^3/24 + (xyz^2)/6 - (xy^2z)/24 + (xyz)/6 \\
-(xy(2x - 6yz + x^2 + 3y^2))/72 \\
-(x^2y^2)/24 - (x^2y)/12 - x^3/36 - x^4/144 - y^4/16 - (x^2yz)/12
\end{array}\right]
$$

3D Printing

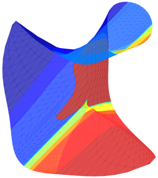
1. Run Bertini

```
****** Witness Set Decomposition ********

| dimension | components | classified
-----------------------------------------
|   2        |   1         |   4
-----------------------------------------
=========
******* Decomposition by Degree *********

Dimension 2: 1 classified component
-----------------------------------------
   degree 4: 1 component

*****************************************
```

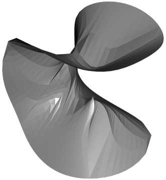
2. Run Bertini_real



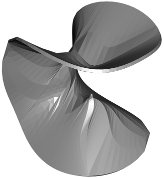
3. Refine



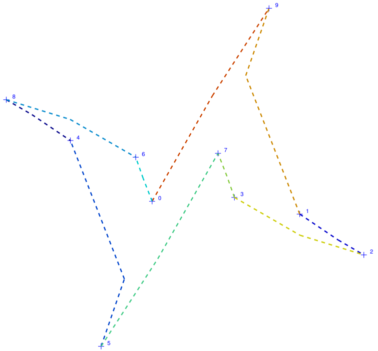
4. Process into `.stl`



5. Thicken surface



6. Print



Thank you for your kind attention.

Acknowledgements

- `Bertini_real` is joint work with Dan Bates, Jon Hauenstein, Wenrui Hao, Charles Wampler, and Andrew Sommese.
- We have been graciously funded by the NSF, and DARPA.