Numerical challenges to decomposition of real algebraic surfaces

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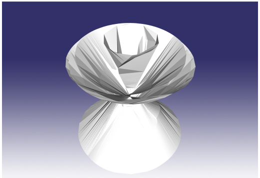
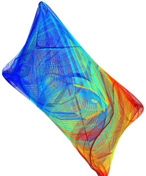
Andrew Sommese, and Charles Wampler

16 January, 2015





Polynomial Systems

Polynomials are everywhere!

- chemical reaction networks
- kinematics
- dynamical systems
- optimization
- many more

example p(z)

$$
\begin{aligned}
x_0 + x_1 + x_2 + y_1 + y_2 + y_3 + y_4 - k_1 &= 0, \\
e + y_1 + y_2 - k_2 &= 0, \\
f + y_3 + y_4 - k_3 &= 0, \\
-a_1 x_0 e + b_1 y_1 + c_4 y_4 &= 0, \\
c_2 y_2 - a_3 x_2 f + b_3 y_3 &= 0, \\
a_1 x_0 e - (b_1 + c_1) y_1 &= 0, \\
a_2 x_1 e - (b_2 + c_2) y_2 &= 0, \\
a_3 x_2 f - (b_3 + c_3) y_3 &= 0, \\
a_4 x_1 f - (b_4 + c_4) y_4 &= 0.
\end{aligned}
$$

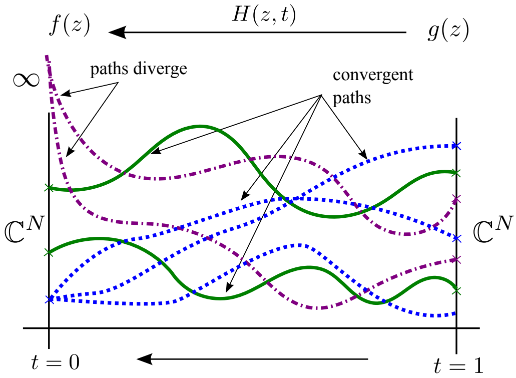
How to solve a system $p(z) = 0$? That's **hard**!

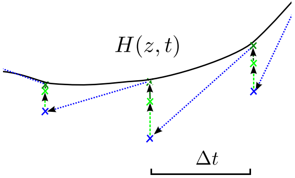
How to solve $p(z) = 0$

There are many ways to 'solve' a polynomial system:

- If linear, invert the matrix. This is just linear algebra.
  Otherwise...
- Symbolically, the main method to use is **Gröbner basis**.
  Available in Macaulay2, Singular, Maple, etc.
  Work exactly over $\mathbb{C}$ to find an ideal basis for the system p.
  $\Rightarrow$ Result is a set of polynomials.
- Numerically, **homotopy continuation** methods find approximate solutions.
  Available in HOM4PS, Bertini, and others
  $\Rightarrow$ Result is a set of points.

Homotopy Methods





Start Systems

There are lots of ways to make start system g.

- Roots of unity - $g_i = z_i^{d_i} - 1$
- Total degree - $g_i = \prod_{j=1}^{d_i} \mathcal{L}_j(\vec{x})$
- Multi-homogeneous - $g_i = \prod_j \mathcal{L}(\vec{x}_{\sigma_k})$
- Polyhedral - g_i determined by the Newton Polytope

The canonical linear homotopy:

$$H(z,t) = t \cdot f(z) + \gamma(1-t) \cdot g(z) = 0$$

Path Tracking

Given a homotopy $H(z,t)$, we wish to move in t from t_1 to t_0. How? By solving the Davidenko differential equation:

$$\frac{\partial H(z(t),t)}{\partial t} + \sum_{i=1}^{N} \frac{\partial H(z(t),t)}{\partial z_i} \frac{dz_i(t)}{dt} = 0$$

which more or less amounts to solving

$$\frac{dz(t)}{dt} = -[JH(z(t),t)]^{-1} \frac{\partial H(z(t),t)}{\partial t}.$$

Path Tracking

We can solve this diff-eq by discrete approximation and an Euler prediction step:

$$p_{i+1} = p_i - [JH(p_i, t_i)]^{-1} \frac{\partial H(p_i, t_i)}{\partial t} \Delta t_i$$

followed by a Newton correction step:

$$z_{i+1} = z_i - [JH(z_i, t_{i+1})]^{-1} H(z_i, t_i)$$

Difficulty 1

Our first numerical difficulties:

- How to choose Δt_i?
- Is $[JH(p_i, t_i)]^{-1}$ and $[JH(z_i, t_{i+1})]^{-1}$ always computable?

Solutions to these problems:

- Δt_i is chosen **adaptively**, and we are permitted to step tentatively.
- Use **adaptive multiple precision**, to overcome the loss of precision through inversion.
  $\Rightarrow$ you lose precision when inverting according to the *condition number*, the ratio of the two extreme singular values.

Variety

The *zero-set* of a system of polynomials is the **variety**.
Varieties consist of **components**, which have meta-data attached to them:

- dimension,
- degree,
- and others.

One primary goal of algebraic geometry is to discover, for $p(z)$, what is the structure of its variety $\mathcal{V}(p)$?

Witness Sets

Witness set - Numerical Algebraic Geometry's representation of a positive dimensional component C over $\mathbb{C}^N$.

Consists of three things:

- points x – generic, nonsingular. $\# = \text{degree}(C)$.
- linears L – generic. $\# = \dim(C)$.
- system f – possibly randomized, Jf is full rank on each x.

Positive dimensional $\mathbb{C}$ components are nice - degree and structure are same almost everywhere.

$\mathbb{R}$ not so much...

Real Posdim Components

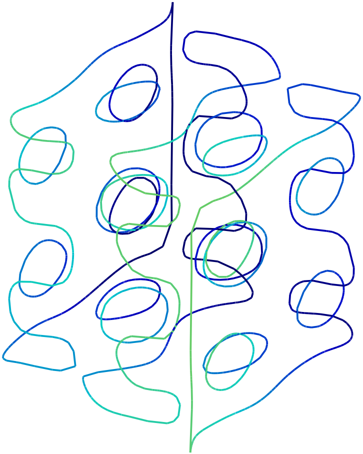
Bertini_real – Numerical Cellular Decomposition

Setup

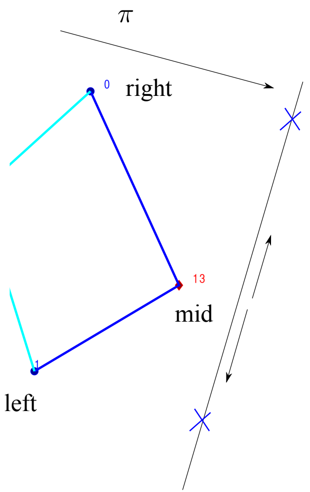
- Let f be a polynomial system with $\mathbb{R}$ coefficients, and $f : \mathbb{C}^N \to \mathbb{C}^n$.
- Let $\mathcal{V}(f)$ be the variety of f.
- Consider $C \subseteq \mathcal{V}(f)$ be a component of dimension k.

Our objective is to decompose the real part of C; i.e., $C \cap \mathbb{R}^N$ or $\mathcal{V}_{\mathbb{R}}(f)$.

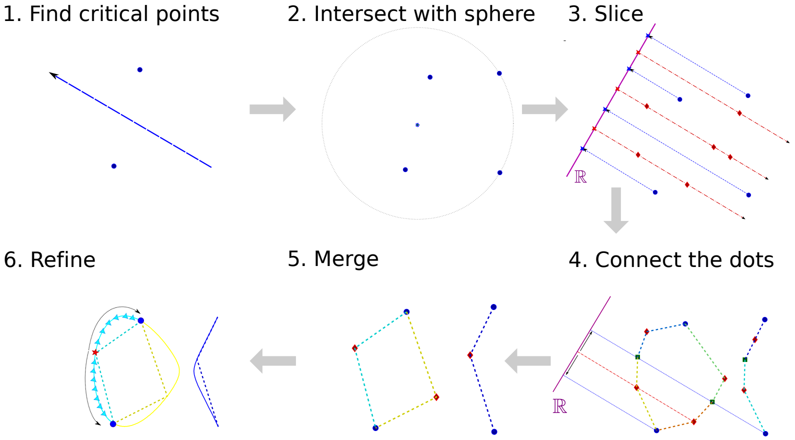
Preview



Curve Cell

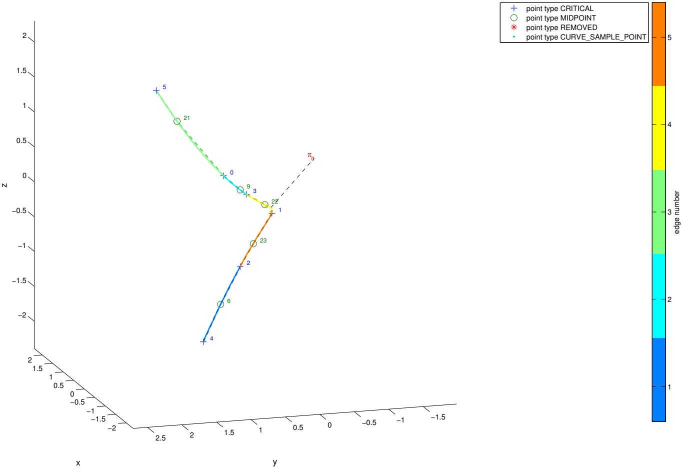


Curves



[Lu,Bates,Sommese,Wampler,2006]

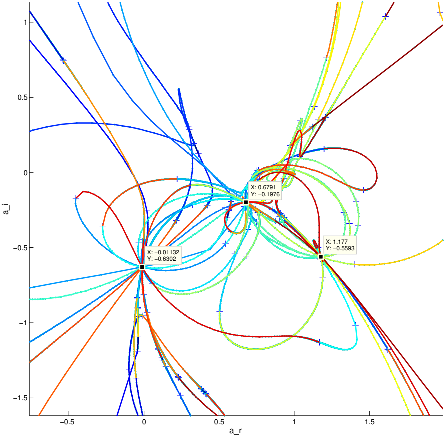
Curves



twisted cubic

$$f(x,y,z) = \begin{bmatrix} y - x^2 \\ z - x^3 \\ y^2 - xz \end{bmatrix}$$

Curves



Burmester 3-3 curve, dimension 14.
In 2d projection, of degree 128.

Critical points of curves

Computing critical points of curves is easy.[*citation needed*]

Since f defines a dimension-one component, the working witness set comes with one random linear:

$$\begin{bmatrix} f \\ \mathcal{L}_1 \end{bmatrix}$$

Then we use **regeneration** to solve the system

$$\begin{bmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \end{pmatrix} \end{bmatrix}$$

[Hauenstein,Sommese,Wampler, 2009]

Numerical Challenge 2

The second challenge –

- critical points are often **singular**.

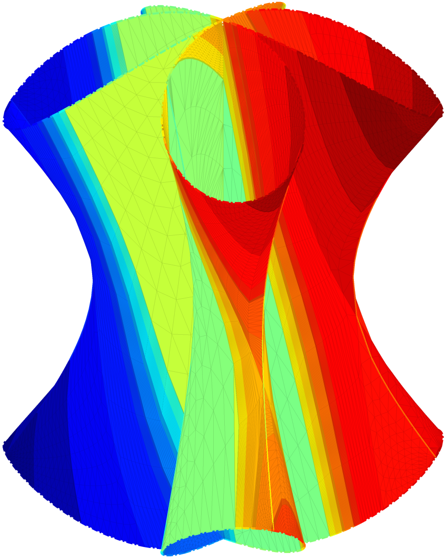
That is, the Jacobian is singular! The condition number is infinite!

The solution –

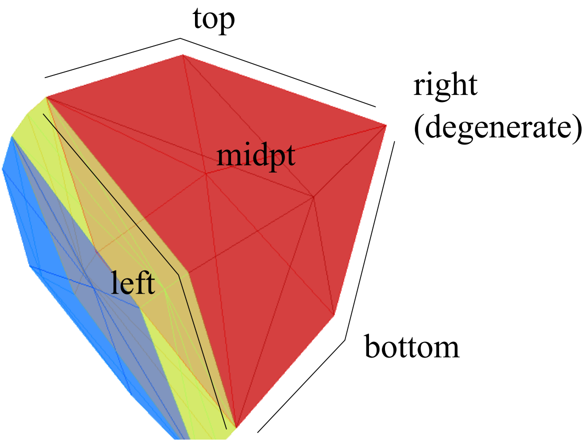
- compute nullvectors simultaneously to the z variables of interest:

$$\begin{bmatrix} f(x) \\ v^{\intercal} \cdot \begin{bmatrix} Jf(x) \\ J\pi_1 \end{bmatrix} \\ \text{patch}_v \end{bmatrix}$$

Surfaces

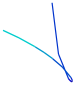


Surface Cell

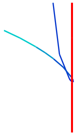


Surface Decomposition

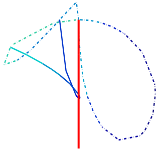
1. Decompose critical curve



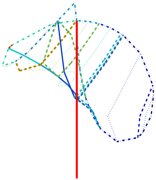
2. Decompose singular curves



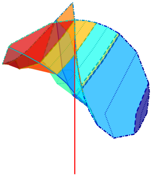
3. Intersect with sphere



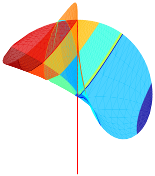
4. Slice



5. Connect the dots

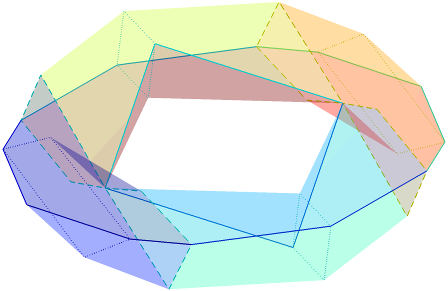


6. Refine



[Besana,Di Rocco,Hauenstein,Sommese,Wampler, 2013]

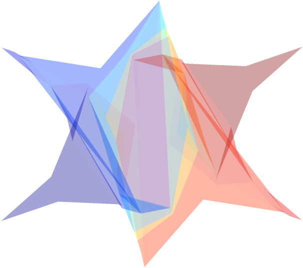
Surface examples



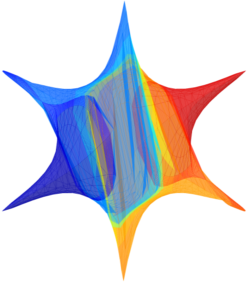
torus

$$f(x,y,z) = (x^2 + y^2 + z^2 + 2^2 - \frac{1}{2}^2)^2 - 16(x^2 + y^2)$$

Surface examples



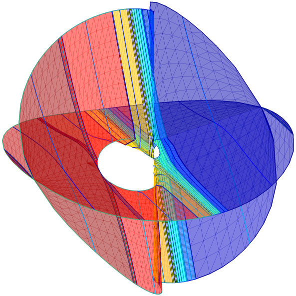
distel, unrefined



distel, refined

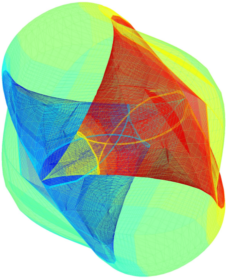
$$f(x,y,z) = x^2 + y^2 + z^2 + 1000(x^2+y^2)(x^2+z^2)(y^2+z^2) - 1$$

Surface examples



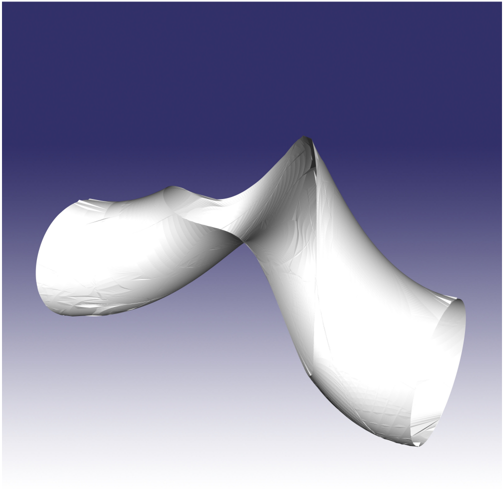
solitude

$$f(x,y,z) =$$
$$x^2yz + xy^2 + y^3 + y^3z - x^2z^2$$



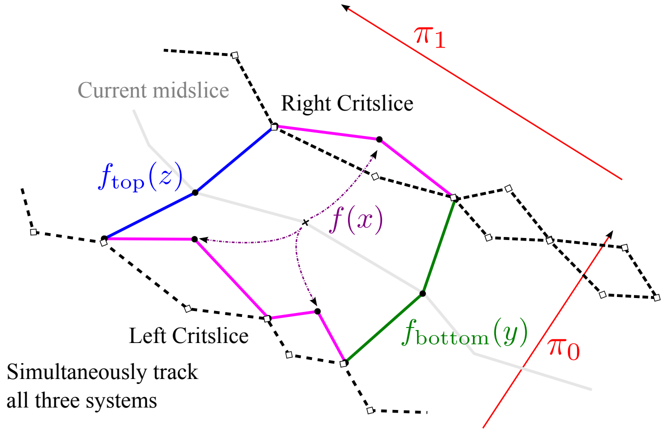
klein

$$f(x,y,z,w) =$$
$$\begin{bmatrix} w^2 + x^2 + y^2 + z^2 - 1 \\ 2wyz - x(y^2+z^2) \end{bmatrix}$$



Fivebar mechanism kinematics
Six-dimensional trigonometric system,
passed though an atan2 projection

Midtracking



Challenge 3

Our third challenge:

- Midtracking tracks *three* systems simultaneously,
- The two boundary systems are often critical or singular curves,
- The endpoints on the boundary are often singular.

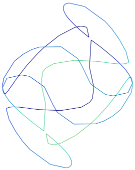
Solution:

- Rely on *adaptive multiple precision* and endgames to keep things nice.

Crit points of surfaces

Computing critical points of surfaces is hard.

- Surfaces have *critical curves*.
- Surfaces also have critical points themselves.



Numerical Challenge 4

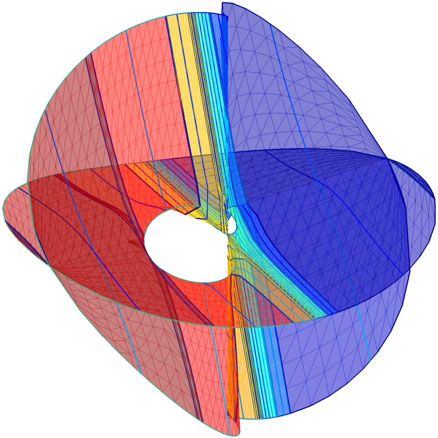
Using the *determinantal* form of the criticality conditions:

witness set:

$$f_{\text{crit\_curve}} = \begin{bmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \\ J\pi_2 \end{pmatrix} \\ \mathcal{L}_1 \end{bmatrix} \qquad f_{\text{crit\_crit}} = \begin{bmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \\ J\pi_2 \end{pmatrix} \\ \det \begin{pmatrix} J \begin{pmatrix} f \\ \det \begin{pmatrix} Jf \\ J\pi_1 \\ J\pi_2 \end{pmatrix} \end{pmatrix} \\ J\pi_1 \end{pmatrix} \\ \text{patch}_v \end{bmatrix}$$

- Determinant operation produces high degree polynomials, contributing to numerical issues.
- Using this formulation for dimension 3 decompositions will be even worse w.r.t. degree and computational complexity.
- Fortunately, we can still use the nullspace method for finding critical points of this curve.

Example - Solitude



How to find singular curves

1. Compute witness set for critical curve.
2. Separate singular points from nonsingular.
   - Nonsingular $\rightarrow$ critical curve.
   - Singular $\rightarrow$ singular curve(s).
3. Separate singular witness points by deflation sequence.
4. Decompose each singular curve.

Practical Challenge 5 - Deflation

Deflation

- Making the untrackable, trackable.
- Removing singularity of points.

Deflation:

- Determinental - adds no variables, but **lots** of functions

$$g_{\text{det}} = \begin{bmatrix} f(x) \\ \det J_{\sigma_1} f(x) \\ \vdots \\ \det J_{\sigma_m} f(x) \end{bmatrix}$$

[Hauenstein, Wampler '13]

Example - Solitude

Solitude:

$$f(x,y,z) = x^2yz + xy^2 + y^3 + y^3z - x^2z^2$$

$$\mathcal{D}^1 = \begin{bmatrix} x^2yz + xy^2 + y^3 + y^3z - x^2z^2 \\ \\ y^2/4 - (xz^2)/2 + (xyz)/2 \\ (xy)/2 + (x^2z)/4 + (3y^2z)/4 + (3y^2)/4 \\ (x^2y)/4 - (x^2z)/2 + y^3/4 \end{bmatrix}$$

Example - Solitude

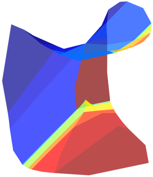
$$\mathcal{D}^2 = \begin{bmatrix} x^2yz + xy^2 + y^3 + y^3z - x^2z^2 \\ \\ y^2/4 - (xz^2)/2 + (xyz)/2 \\ (xy)/2 + (x^2z)/4 + (3y^2z)/4 + (3y^2)/4 \\ (x^2y)/4 - (x^2z)/2 + y^3/4 \\ \\ (y^2z^2)/8 - (x^2z^3)/24 + (y^2z^3)/8 - (y^3z^2)/8 - (y^3z)/8 + y^3/24 + (xy^2z)/24 + (x^2yz^2)/24 \\ (x^2z^3)/12 + (y^3z^2)/24 + (xy^3)/24 - (y^4z)/24 - (xy^2z)/12 - (x^2yz^2)/8 + (x^2y^2z)/24 \\ (x^2y^2)/24 + (xy^3)/8 - y^4/24 - (xy^2z)/4 - (x^2yz)/12 + (xy^3z)/12 - (xy^2z^2)/4 \\ -(x(x^2z^2 - 3y^2z^2 + 2xz^2 + 6yz^2 - 3y^2z + 6yz^3 + y^2 + xyz))/24 \\ (x^3z^2)/24 - (x^2y^2)/48 + y^4/48 + (x^2yz)/12 + (xy^3z)/12 - (xy^2z^2)/8 \\ (xy^3)/12 - (x^2y^2)/16 + (x^3z)/12 + (x^4z)/48 + (y^4z)/16 + y^4/16 + (x^2yz)/4 + (x^2yz^2)/4 \\ (x^3z^2)/24 - (x^2y^2)/16 - (xy^3)/8 + y^4/16 + (xy^2z)/4 + (x^2yz)/6 + (xy^2z^2)/8 \\ -(x(x^2y^2 + 2x^2z^2 + xy^2 - 2y^3z + y^4 - 2x^2yz))/24 \\ -(y(4x^2y^2 + 6x^2y + 4x^3 + x^4 + 3y^4))/48 \\ (y^2z^2)/12 - (x^2z^2)/36 - (xz^2)/36 - (yz^2)/12 + (y^2z)/12 - (yz^3)/12 - y^2/36 - (xyz)/36 \\ (x^2z^2)/24 - (y^2z^2)/24 - (xy^2)/36 + (y^3z)/24 - (x^2yz)/72 + (xyz)/18 \\ (x^2z)/18 - (x^2y)/72 - (xy^2)/12 + (x^3z)/72 + y^3/24 + (xyz^2)/6 - (xy^2z)/24 + (xyz)/6 \\ (x^2z^2)/24 - (y^2z^2)/24 - (xy^2)/36 + (y^3z)/24 - (x^2yz)/72 + (xyz)/18 \\ -(x^2(y^2 - 3yz + 3z^2))/36 \\ -(xy(2x - 6yz + x^2 + 3y^2))/72 \\ (x^2z)/18 - (x^2y)/72 - (xy^2)/12 + (x^3z)/72 + y^3/24 + (xyz^2)/6 - (xy^2z)/24 + (xyz)/6 \\ -(xy(2x - 6yz + x^2 + 3y^2))/72 \\ -(x^2y^2)/24 - (x^2y)/12 - x^3/36 - x^4/144 - y^4/16 - (x^2yz)/12 \end{bmatrix}$$

3D Printing

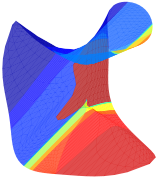
1. Run Bertini

```
****** Witness Set Decomposition ********

| dimension | components | classified
-----------------------------------------
|   2       |    1       |    4
-----------------------------------------
=========
******* Decomposition by Degree *********

Dimension 2: 1 classified component
-----------------------------------------
   degree 4: 1 component

*****************************************
```

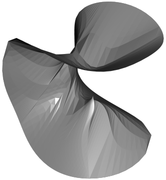
2. Run Bertini_real



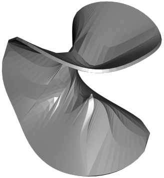
3. Refine



4. Process into `.stl`



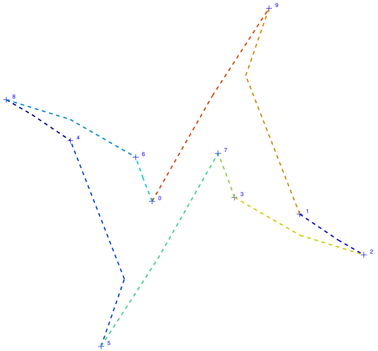
5. Thicken surface



6. Print

Last Notes

- There are lots of ways to get ahold of algebraic varieties,
- I discussed *numerical* methods here,
- The fundamental computed unit at every stage is *a point*,
- Inversion of matrices incurs *loss of precision*,
- Techniques to overcome numerical challenges involve:
  - Solving a different problem (*reformulation*),
  - Using longer numbers (*adaptive multiple precision*),
  - Using fancy techniques at the end of a path (*endgames*),

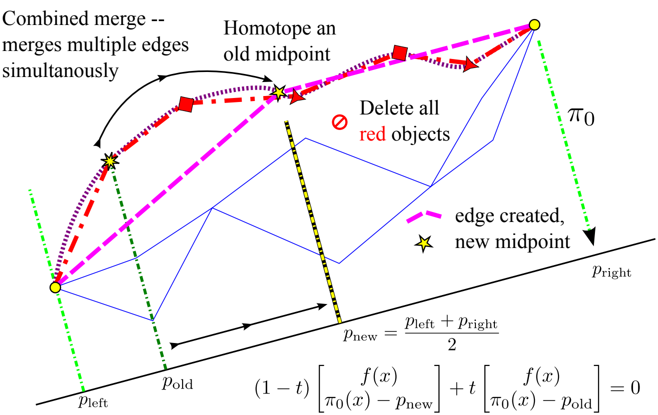


Thank you for your kind attention.

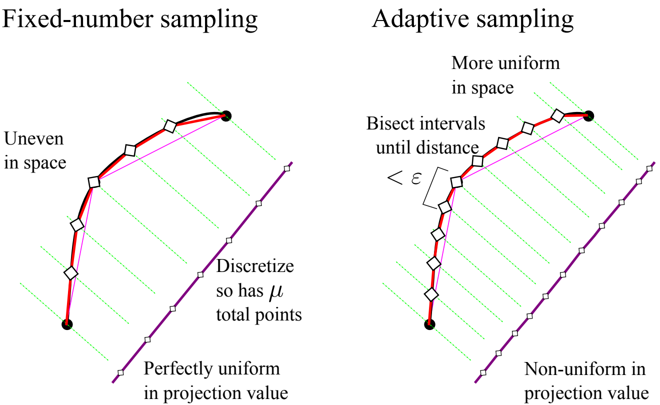
Acknowledgements

- `Bertini_real` is joint work with Dan Bates, Jon Hauenstein, Wenrui Hao, Charles Wampler, and Andrew Sommese.
- We have been graciously funded by the NSF, and DARPA.

Curve Merging

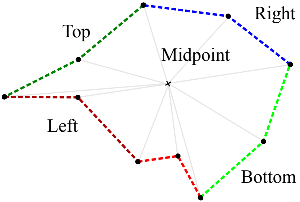


Curve Sampling

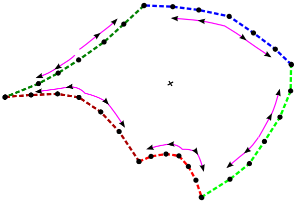


Surface Refinement

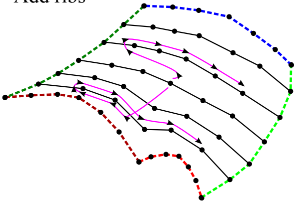
Raw face and edges



Refine the edges



Add ribs



Triangulate